You may cite results from the outline or the exercises without proof. State your citation clearly, e.g., as Theorem/Proposition x from the outline, or as Exercise y, Question z.
Supply reasons for all other assertions.

In this paper, λ denotes Lebesgue measure on $\mathbb{R}$.

Question 1 [10 points] Let $(\varepsilon_k)_{k=1}^{\infty}$ be a sequence of real numbers such that $\varepsilon_k > 0$ for all k and $\sum_{k=1}^{\infty} \varepsilon_k < \infty$. Assume that a sequence of finite-valued measurable functions $(f_k)_{k=1}^{\infty}$ on a measure space (Ω, Σ, μ) satisfies

$$\mu\{|f_k - f_{k+1}| \geq \varepsilon_k\} \leq \varepsilon_k \text{ for all } k.$$

Show that there exists a set $E \in \Sigma$ such that $\mu(E^c) = 0$ and that $(f_k(\omega))_{k=1}^{\infty}$ converges to a finite value for all $\omega \in E$.

Question 2 [10 points] Let $(f_n)_{n=1}^{\infty}$ be a sequence of Lebesgue measurable functions on $\mathbb{R}$ such that $\sup_n \|f_n\|_\infty < \infty$. Suppose that for any pair of real numbers $a < b$, $\lim_{n\to\infty} \int_{[a,b]} f_n \, d\lambda = 0$. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n g \, d\lambda = 0 \text{ for each Lebesgue integrable function } g \text{ on } \mathbb{R}.$$

Question 3 [10 points] Let f be a real-valued Lebesgue integrable function on $\mathbb{R}$ so that the function g defined by $g(x) = |x|^{3/2} f(x)$, $x \in \mathbb{R}$, is also Lebesgue integrable on $\mathbb{R}$. Set

$$h(t) = \int_{\mathbb{R}} f(x) \sin(tx) \, d\lambda(x) \text{ for all } t \in \mathbb{R}.$$

Show that h is differentiable on $\mathbb{R}$ and that there is a finite constant C such that

$$|h'(t) - h'(s)| \leq C|t - s|^{1/2} \text{ for all } t, s \in \mathbb{R}.$$

Question 4 [10 points] Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function that is Lebesgue integrable on $[0,\infty)$. For any $x \in [0,\infty)$, define $g_x : [1,\infty) \to \mathbb{R}$ by

$$g_x(y) = \frac{3x^2 + y}{y^2} f(x^3 + xy + y^2).$$

Show that g_x is Lebesgue integrable on $[1,\infty)$ for λ-almost all $x \in [0,\infty)$.

Question 5 [10 points] Let (Ω, Σ, μ) be a measure space such that $\mu(\Omega) < \infty$. Suppose that $1 < p < \infty$ and that $(f_n)_{n=1}^{\infty}$ is a sequence of real-valued functions in $\mathcal{L}^p(\Omega, \Sigma, \mu)$ with $\sup_n \|f_n\|_p < \infty$.

(a) Show that the set $\{f_n : n \in \mathbb{N}\}$ is uniformly integrable with respect to μ.

(b) Assume that $(f_n)_{n=1}^{\infty}$ converges in measure to a real-valued μ-measurable function f. Show that $f \in \mathcal{L}^1(\Omega, \Sigma, \mu)$ and that $\lim_{n\to\infty} \|f_n - f\|_1 = 0$.

Question 6 [10 points] Let f be a Lebesgue integrable function on $\mathbb{R}$. Define $g : \mathbb{R} \to [0,\infty]$ by

$$g(t) = \lim_{r\to 0+} \sup\left\{\frac{\int_a^b |f|\, d\lambda}{b-a} : t - r < a < t < b < t + r\right\}.$$

Show that for all $c > 0$,

$$c\lambda^*(\{g > c\}) \leq \int |f|\, d\lambda, \text{ where } \lambda^* \text{ denotes Lebesgue outer measure.}$$

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

MA5205 – Graduate Analysis I

(Semester 1 : AY2014/15)

Time allowed : 2 hours 30 minutes

INSTRUCTIONS TO STUDENTS

1. Please write your matriculation/student number only. Do not write your name.
2. This examination paper contains a total of **SIX (6)** questions and comprises **THREE (3)** printed pages.
3. Students are required to answer **ALL** questions. The maximum score for this examination is 60 points.
4. Please start each question on a new page.
5. This is a CLOSED BOOK examination. Students are allowed to bring the notes made available on IVLE.
6. Students may use calculators.