Matriculation Number:

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2012-2013

MA1506 MATHEMATICS II

April 2013 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
(a)								
(b)								

Question 1 (a) [5 marks]

True or False question.

Consider the first order ordinary differential equation

$$\frac{dy}{dx} = \frac{x}{y}.$$

For each of the following parts, answer **T** if you think the statement is true and answer **F** if you think it is false.

(i) It is a separable equation.

(ii) It is a linear equation.

Answer 1(a)(i)	T	**Answer 1(a)(ii)**	F

(Show your working below and on the next page.)

Question 1 (b) [5 marks]

You invented a new gadget which you wanted to sell in Singapore. You planned to introduce your gadget through an advertising campaign to the population of five million there. The rate at which the population hears about the product is assumed to be proportional to the number of people who have not yet heard of the product. A survey done one month after you have started your campaign found that one million people have heard of your product. Assume that nobody knew about your product before you started your campaign and that you planned to run the campaign up to the time when three million people have heard of your product, how many months after you have started your campaign should you stop the campaign? Give your answer correct to one decimal place.

Answer 1(b)	4.1

(Show your working below and on the next page.)

Let N million = number of people who have heard of the product at time t months

$$\frac{dN}{dt} = k(5-N)$$

$$\frac{dN}{5-N} = k\,dt \Rightarrow -\ln|5-N| = kt + C_1$$

$$\Rightarrow 5-N = Ae^{-kt}$$

$$N(0)=0 \Rightarrow 5 = A$$

$$\therefore N = 5-5e^{-kt}$$

$$N(1)=1 \Rightarrow 1 = 5-5e^{-k} \Rightarrow 5e^{-k} = 4$$

$$\Rightarrow k = -\ln(0.8)$$

$$\therefore N = 5-5e^{t\ln(0.8)}$$

$$N=3 \Rightarrow 5e^{t\ln(0.8)} = 2 \Rightarrow t = \frac{\ln(0.4)}{\ln(0.8)}$$

$$= \underline{\underline{4.1}}06\cdots$$

Question 2 (a) [5 marks]

A particle moves along the x-axis in simple harmonic motion with its displacement x (measured in metres) from the origin at time t (measured in seconds) satisfies the differential equation

$$\ddot{x} + 101x = 0.$$

In addition, it is known that $x(0) = 1$ metre and $\dot{x}(0) = -5$ metre per second. Find the amplitude (measured in metres) of the motion. Give your answer correct to two decimal places.

Answer 2(a)	1.12

(Show your working below and on the next page.)

$\lambda^2 + 101 = 0 \Rightarrow \lambda = \pm i\sqrt{101}$

$\therefore x = C_1 \cos\sqrt{101}\,t + C_2 \sin\sqrt{101}\,t$

$\therefore \dot{x} = -C_1\sqrt{101}\sin\sqrt{101}\,t + C_2\sqrt{101}\cos\sqrt{101}\,t$

$x(0) = 1 \Rightarrow C_1 = 1$

$\dot{x}(0) = -5 \Rightarrow C_2 = -\dfrac{5}{\sqrt{101}}$

$\therefore x = \cos\sqrt{101}\,t - \dfrac{5}{\sqrt{101}}\sin\sqrt{101}\,t$

amplitude $= \sqrt{1^2 + \left(\dfrac{-5}{\sqrt{101}}\right)^2} = \sqrt{\dfrac{126}{101}}$

$= 1.116\cdots$

$\simeq \underline{\underline{1.12}}$

Question 2 (b) [5 marks]

A particle moves along the x-axis in forced oscillation without damping and with its displacement x (measured in metres) from the origin at time t (measured in seconds) satisfies the differential equation

$$\ddot{x} + 100x = 19 \cos \alpha t,$$

where α denotes a positive constant. In addition, it is known that $x(0) = 0$ and $\dot{x}(0) = 0$. Find the value of α when the system has resonance.

Answer 2(b)	10

(Show your working below and on the next page.)

$\lambda^2 + 100 = 0 \Rightarrow \lambda = \pm 10i$

resonance $\Rightarrow \alpha = \underline{\underline{10}}$

Question 3 (a) [5 marks]

A certain fish population was protected by law and its population N (measured in tons of fish) follows a logistic growth model

$$\frac{dN}{dt} = N - \frac{N^2}{12},$$

where time is measured in years. Eventually, the fish population reached its logistic equilibrium and the government removed the protection and allowed a harvesting quota of 1.5 tons of fish per year. What will the fish population (measured in tons of fish) be in the long run after harvesting was allowed? Give your answer correct to two decimal places.

Answer 3(a)	10.24

(Show your working below and on the next page.)

$$N - \frac{N^2}{12} - 1.5 = 0$$

$$\Rightarrow N^2 - 12N + 18 = 0$$

$$N = \frac{12 \pm \sqrt{144 - 72}}{2}$$

$$= 6 \pm 3\sqrt{2}$$

$$\beta_2 = 6 + 3\sqrt{2} = \underline{\underline{10.242\ldots}}$$

Question 3 (b) [5 marks]

After beating Ah Huat Contractor Services and won the HDB balcony building contract, the billionaire engineer Tan Ah Lian rewarded herself by buying a huge bungalow. Before she moved in, she decided to add a balcony to the bungalow. From her good lessons in MA1506 at NUS, she knew that the shape of the balcony satisfied the Euler Equation

$$\frac{d^4y}{dx^4} = \frac{w(x)}{EI},$$

where $w(x)$ is the weight function and E and I are constants. Furthermore, y also satisfied the boundary conditions

$$\frac{d^3y}{dx^3}(L) = \frac{d^2y}{dx^2}(L) = \frac{dy}{dx}(0) = y(0) = 0,$$

where L is the length of the balcony. She chose a design which had $w(x) = -\cos\frac{\pi x}{2L}$, and calculated that the dip at the end of her balcony is $-\frac{cL^4}{EI}$ where c denotes a constant. Find the value of c. Give your answer correct to three decimal places.

Answer 3(b)	0.048

(Show your working below and on the next page.)

$$\frac{d^4y}{dx^4} = -\frac{1}{EI}\cos\frac{\pi x}{2L}$$

$$\frac{d^3y}{dx^3} = -\frac{2L}{\pi EI}\sin\frac{\pi x}{2L} + C_1$$

$$\frac{d^3y}{dx^3}(L) = 0 \Rightarrow C_1 = \frac{2L}{\pi EI}$$

(More working space for Question 3(b))

$$\therefore \frac{d^3y}{dx^3} = -\frac{2L}{\pi EI}\sin\frac{\pi x}{2L} + \frac{2L}{\pi EI}$$

$$\frac{d^2y}{dx^2} = \frac{4L^2}{\pi^2 EI}\cos\frac{\pi x}{2L} + \frac{2Lx}{\pi EI} + C_2$$

$$\frac{d^2y}{dx^2}(L) = 0 \Rightarrow C_2 = -\frac{2L^2}{\pi EI}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{4L^2}{\pi^2 EI}\cos\frac{\pi x}{2L} + \frac{2Lx}{\pi EI} - \frac{2L^2}{\pi EI}$$

$$\frac{dy}{dx} = \frac{8L^3}{\pi^3 EI}\sin\frac{\pi x}{2L} + \frac{Lx^2}{\pi EI} - \frac{2L^2 x}{\pi EI} + C_3$$

$$\frac{dy}{dx}(0) = 0 \Rightarrow C_3 = 0$$

$$y = \frac{-16L^4}{\pi^4 EI}\cos\frac{\pi x}{2L} + \frac{Lx^3}{3\pi EI} - \frac{L^2x^2}{\pi EI} + C_4$$

$$y(0) = 0 \Rightarrow C_4 = \frac{16L^4}{\pi^4 EI}$$

$$\therefore y = -\frac{16L^4}{\pi^4 EI}\cos\frac{\pi x}{2L} + \frac{Lx^3}{3\pi EI} - \frac{L^2x^2}{\pi EI} + \frac{16L^4}{\pi^4 EI}$$

$$x = L \Rightarrow y = \left(\frac{1}{3\pi} - \frac{1}{\pi} + \frac{16}{\pi^4}\right)\frac{L^4}{EI}$$

$$= (-0.0479\cdots)\frac{L^4}{EI} \approx \underline{\underline{-0.048}}\frac{L^4}{EI}$$

Question 4 (a) [5 marks]

(i) Let $F(s)$ denote the Laplace Transform of $e^t \sin^2 t$. Find the exact value of $F(2)$.

(ii) Let $f(t)$ denote the inverse Laplace Transform of

$$\frac{4s^2+11s+9}{(s+1)^2(s+2)} \quad .$$

Find the value of $f(\frac{1}{2})$. Give your answer correct to one decimal place.

Answer 4(a)(i)	$\frac{2}{5}$	**Answer 4(a)(ii)**	2.3

(Show your working below and on the next page.)

(i) $$L(e^t \sin^2 t) = L\left(e^t \frac{1-\cos 2t}{2}\right)$$
$$= \frac{1}{2}L(e^t) - \frac{1}{2}L(e^t \cos 2t)$$
$$= \frac{1}{2(s-1)} - \frac{1}{2}\frac{s-1}{(s-1)^2+2^2}$$
$$s=2 \Rightarrow \frac{1}{2} - \frac{1}{2\{1^2+2^2\}} = \underline{\underline{\frac{2}{5}}}$$

(ii) $$L^{-1}\left[\frac{4s^2+11s+9}{(s+1)^2(s+2)}\right] = L^{-1}\left\{\frac{1}{s+1} + \frac{2}{(s+1)^2} + \frac{3}{s+2}\right\}$$
$$= e^{-t} + 2te^{-t} + 3e^{-2t}$$
$$t = \frac{1}{2} \Rightarrow e^{-\frac{1}{2}} + e^{-\frac{1}{2}} + 3e^{-1}$$
$$= \underline{\underline{2.3}}16\cdots$$

Question 4 (b) [5 marks]

Find the solution of the initial value problem

$$y'' + 2y' + 2y = \delta(t - \pi), \quad y(0) = y'(0) = 0,$$

where δ is the Dirac Delta function. Give your answer in terms of the unit step function.

Answer 4(b)	$y = e^{-(t-\pi)} \sin(t-\pi)\, u(t-\pi)$

(Show your working below and on the next page.)

$$L(y''+2y'+2y) = L(\delta(t-\pi))$$

$$\Rightarrow s^2 L(y) - s y(0) - y'(0)$$

$$+ 2(sL(y) - y(0)) + 2L(y) = e^{-\pi s}$$

$$s^2 Y + 2sY + 2Y = e^{-\pi s} \quad \text{where } Y = L(y)$$

$$Y = \frac{e^{-\pi s}}{s^2+2s+2}$$

$$= \frac{1}{(s+1)^2 + 1^2} e^{-\pi s}$$

Recall that $L^{-1}\left(\frac{1}{(s+1)^2+1^2}\right) = e^{-t} \sin t$

$$\therefore y = \underline{\underline{e^{-(t-\pi)} \sin(t-\pi)\, u(t-\pi)}}$$

Question 5 (a) [5 marks]

Let $M = \begin{pmatrix} 7 & -12 \\ 4 & -7 \end{pmatrix}$. Find the exact value of M^{2013}.

(Suggestion: find the matrix M^2 first.)

Answer 5(a)	$\begin{pmatrix} 7 & -12 \\ 4 & -7 \end{pmatrix}$

(Show your working below and on the next page.)

$$M^2 = \begin{pmatrix} 7 & -12 \\ 4 & -7 \end{pmatrix}\begin{pmatrix} 7 & -12 \\ 4 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore M^{2013} = (M^2)^{1006} M$$

$$= M = \underline{\underline{\begin{pmatrix} 7 & -12 \\ 4 & -7 \end{pmatrix}}}$$

Question 5 (b) [5 marks]

Find the matrix of a shear [shearing angle 45 degrees] when the shearing forces are parallel to an axis which makes an angle of 45 degrees with the x-axis. Give exact values for your answer.

Answer 5(b)	$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$

(Show your working below and on the next page.)

$R(\theta)$ = rotate counter-clockwise θ

$S(\theta)$ = shear θ // x-axis

$$M = R(45^\circ)\,S(45^\circ)\,R(-45^\circ)$$

$$= \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \begin{pmatrix} 1 & \tan 45^\circ \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

Question 6 (a) [5 marks]

S and T are two 3×3 matrices that satisfy the equation

$$T = 1506S - 501(\mathrm{Tr}(S))I \quad ,$$

where $\mathrm{Tr}(S)$ denotes the trace of S and I denotes the 3×3 identity matrix. If

$$T = \begin{pmatrix} 28 & 55 & 63 \\ 87 & 46 & 19 \\ 32 & 97 & 76 \end{pmatrix} \quad ,$$

find the exact value of $\mathrm{Tr}(S)$.

Answer 6(a)	50

(Show your working below and on the next page.)

$$\mathrm{Tr}(T) = 1506\,\mathrm{Tr}(S) - 501(\mathrm{Tr}\,S)(\mathrm{Tr}\,I)$$
$$= 3\,\mathrm{Tr}(S)$$
$$\therefore \mathrm{Tr}(S) = \tfrac{1}{3}\,\mathrm{Tr}(T)$$
$$= \tfrac{1}{3}(150)$$
$$= \underline{\underline{50}}$$

Question 6 (b) [5 marks]

A company has two branches making products A and B respectively. Branch A consumes \$ 0.2 of the B output and \$ 0.5 of its own output for every \$ 1 it produces. Branch B consumes \$ 0.6 of the A output and \$ 0.4 of its own output for every \$ 1 it produces. The company needs to meet a yearly external demand of 5 million dollars for its A product and 4 million dollars for its B product. What is the dollar value of (i) its A product and (ii) its B product that the company should produce per year? Give exact value in millions of dollars for your answers.

Answer 6(b)(i) A =	30	Answer 6(b)(ii) B =	$\frac{50}{3}$

(Show your working below and on the next page.)

$$\begin{cases} A = 0.5A + 0.6B + 5 \\ B = 0.2A + 0.4B + 4 \end{cases}$$

$$\begin{pmatrix} 0.5 & -0.6 \\ -0.2 & 0.6 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 30 \\ \frac{50}{3} \end{pmatrix}$$

Question 7 (a) [5 marks]

True or False question.

For each of the following parts, answer **T** if you think the statement is true and answer **F** if you think it is false.

(i) If the value of the constant c satisfies $c > \frac{1}{2}$, then the following system of differential equations represents a nodal source.

$$\begin{cases} \frac{dx}{dt} = 2x - cy \\ \frac{dy}{dt} = cx + y \end{cases}.$$

(ii) The following system of differential equations represents a centre.

$$\begin{cases} \frac{dx}{dt} = -8x + 13y \\ \frac{dy}{dt} = -5x + 8y \end{cases}.$$

Answer 7(a)(i)	F	Answer 7(a)(ii)	T

(Show your working below and on the next page.)

(i) We have to check three conditions:

one: $\begin{vmatrix} 2 & -c \\ c & 1 \end{vmatrix} = 2 + c^2 > 0$ ✓

two: $Tr = 2+1 > 0$ ✓

three: $(Tr)^2 - 4(det) > 0 \Leftrightarrow 9-8-4c^2 > 0$
$\Leftrightarrow |c| < \frac{1}{2}$ X

(ii) Check two conditions:

one: $Tr = 0$ ✓

two $(Tr)^2 - 4(det) = -4 < 0$ ✓

Question 7 (b) [5 marks]

Solve the following system of differential equations:

$$\begin{cases} \frac{dx}{dt} = -2x + y \\ \frac{dy}{dt} = 2x - 3y \end{cases},$$

with initial conditions $x(0) = 1$ and $y(0) = 3$.

Answer 7(b)	$\begin{cases} x = \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t} \\ y = \frac{5}{3}e^{-t} + \frac{4}{3}e^{-4t} \end{cases}$

(Show your working below and on the next page.)

Solution 1

$$\begin{vmatrix} -2-\lambda & 1 \\ 2 & -3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 5\lambda + 4 = 0$$

$$\Rightarrow \lambda = -1, \text{ or } -4$$

$\lambda = -1 \Rightarrow \quad -x + y = 0 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigenvector

$\lambda = -4 \Rightarrow \quad 2x + y = 0 \Rightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ eigenvector

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 - 2c_2 \end{pmatrix} \Rightarrow c_1 = \frac{5}{3}, \quad c_2 = -\frac{2}{3}$$

$$\begin{cases} x = \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t} \\ y = \frac{5}{3}e^{-t} + \frac{4}{3}e^{-4t} \end{cases}$$

Solution 2 Using Laplace Transform.

Let $L(x) = X$, $L(y) = Y$.

$$\begin{cases} sX - x(0) = -2X + Y \\ sY - y(0) = 2X - 3Y \end{cases}$$

$$\begin{cases} (s+2)X - Y = 1 \\ -2X + (s+3)Y = 3 \end{cases}$$

$$\begin{vmatrix} s+2 & -1 \\ -2 & s+3 \end{vmatrix} = s^2 + 5s + 6 - 2 = s^2 + 5s + 4$$
$$= (s+1)(s+4)$$

$$\begin{pmatrix} s+2 & -1 \\ -2 & s+3 \end{pmatrix}^{-1} = \frac{1}{(s+1)(s+4)} \begin{pmatrix} s+3 & 1 \\ 2 & s+2 \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{(s+1)(s+4)} \begin{pmatrix} s+3 & 1 \\ 2 & s+2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \dfrac{s+6}{(s+1)(s+4)} \\ \dfrac{3s+8}{(s+1)(s+4)} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{5}{3(s+1)} - \dfrac{2}{3(s+4)} \\ \dfrac{5}{3(s+1)} + \dfrac{4}{3(s+4)} \end{pmatrix}$$

$$\therefore \begin{cases} x = \frac{5}{3} e^{-t} - \frac{2}{3} e^{-4t} \\ y = \frac{5}{3} e^{-t} + \frac{4}{3} e^{-4t} \end{cases}$$

Question 8 (a) [5 marks]

Use the method of separation of variables to find $u(x,y)$ that satisfies the partial differential equation

$$u_{xy} = [\sin(y)\cos(x)]\,u\ ,$$

given that $u\left(\frac{\pi}{2},\frac{\pi}{2}\right) = e$ and $u\left(-\frac{\pi}{2},-\frac{\pi}{2}\right) = e^3$.

Answer 8(a)	$u = e^{2-\sin x+\cos y}$

(Show your working below and on the next page.)

$$u = XY$$

$$X'Y' = \sin y \cos x\, XY$$

$$\frac{X'}{X\cos x} = \frac{Y\sin y}{Y'} = k$$

$$\frac{X'}{X\cos x} = k \Rightarrow X = Ae^{k\sin x}$$

$$\frac{Y'}{Y\sin y} = \frac{1}{k} \Rightarrow Y = Be^{-\frac{1}{k}\cos y}$$

$$\therefore u = Ce^{k\sin x - \frac{1}{k}\cos y}$$

$$\left.\begin{array}{l} u(\frac{\pi}{2},\frac{\pi}{2}) = e \Rightarrow e = Ce^{k} \\ u(-\frac{\pi}{2},-\frac{\pi}{2}) = e^3 \Rightarrow e^3 = Ce^{-k} \end{array}\right\} \Rightarrow k=-1,\ C=e^2$$

$$\therefore \underline{\underline{u = e^{2-\sin x+\cos y}}}$$

Question 8 (b) [5 marks]

Solve the heat equation

$$u_t = u_{xx},$$

with the boundary conditions

$$u(0,t) = u(\pi,t) = 0 \text{ for all time } t,$$

and initial condition $u(x,0) = f(x)$, where

$$f(x) = \begin{cases} x, \text{ if } 0 \leq x \leq \frac{\pi}{2} \\ \pi - x, \text{ if } \frac{\pi}{2} \leq x \leq \pi \end{cases},$$

with $f(x)$ extended to be an odd function of period 2π. You only need to give the first two non-zero terms in your infinite series solution of u.

(Suggestion: You may want to use the fact that the solution of the heat equation together with the boundary conditions as given in this question is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx,$$

where the b_n's are constants.)

Answer 8(b)	$u = \frac{4}{\pi} e^{-t} \sin x - \frac{4}{9\pi} e^{-9t} \sin 3x + \cdots$

(Show your working below and on the next page.)

(More working space for Question 8(b))

b_n are the coefficients of the sine half range expansion of $f(x)$

$$\therefore b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin nx\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}(\pi - x)\sin nx\,dx$$

$$= \frac{4\sin\frac{n\pi}{2}}{\pi n^2}$$

$$b_1 = \frac{4}{\pi},\ b_2 = 0,\ b_3 = \frac{-4}{9\pi}$$

$$\therefore u = \frac{4}{\pi}e^{-t}\sin x - \frac{4}{9\pi}e^{-9t}\sin 3x + \cdots$$

END OF PAPER