NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 2 EXAMINATION 2011-2012

MA3501 MATHEMATICAL METHODS IN ENGINEERING

April/May 2012 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **NINE (9)** printed pages.

2. Answer **ALL** questions in the examination paper. Marks for each question are indicated at the beginning of the question. The maximum score for this examination is **80 marks.**

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

4. Two A4 handwritten double-sided helpsheets are allowed.

5. This is a **CLOSED BOOK** examination.

Answer all the questions.

Question 1 [20 marks]

(a) The weight of a randomly chosen grape of a given variety may be taken to be a normal variable with mean $5g$. Suppose that the probability of a randomly chosen grape that weighs less than $3g$ is 0.123.

(i) Find the standard deviation, in grams. Give your answer correct to two places of decimals.

(ii) Find the probability that a randomly chosen grape weighs more than $9g$. Give your answer correct to four places of decimals.

(iii) Using a suitable approximation, find the probability that, out of 100 randomly chosen grapes, at least three weigh more than $9g$ each. Give your answer correct to four places of decimals.

(b) A questionnaire was sent to a large number of people, asking for their opinions about a proposal to alter an examination syllabus. Of the 180 replies received, 134 were in favour of the proposal. Assuming that the people replying were a random sample from the population,

(i) test, at the 5% level, the hypothesis that the population proportion in favour of the proposal is 0.7 against the alternative that it is more than 0.7;

(ii) find a symmetric 95% confidence interval for the population proportion in favour of the proposal.

(c) Solve, by the method of characteristics, the following first order partial differential equation.

$$u_x(x,y) + 2xy^2 u_y(x,y) = 0,$$
$$u(1,y) = \sin y.$$

Question 2 [20 marks]

(a) Let A be a 2×2 matrix with real and distinct eigenvalues λ_1 and λ_2. Suppose u and v are the respective eigenvectors. Prove that u and v are linearly independent.

(b) Consider two tanks, each containing 100 litres of water. Both tanks initially contain some amount of salt dissolved in the water. Pure water is poured into tanks A and B at a constant rate of 1 litre per minute for each tank. The thoroughly mixed solution from tank A is constantly pumped into tank B at a rate of 1 litre per minute while the solution from tank B is pumped back to tank A at a rate of 2 litres per minute. The solution in tank A is also pumped out and discarded at a rate of 2 litres per minute. Let $x_1(t)$ and $x_2(t)$ be the amount of salt in Tank A and Tank B, respectively. If $x_1(0) > x_2(0) > 0$,

(i) prove that $x_1(t) > x_2(t)$ for all $t > 0$;

(ii) evaluate $\lim_{t\to\infty} \frac{x_1(t)}{x_2(t)}$. Justify your answer.

(c) Consider the boundary value problem

$$\begin{aligned} u_{xx}(x,t) &= u_t(x,t) + 2x, && 0 < x < 1,\ t > 0, \\ u(x,0) &= x, && 0 < x < 1, \\ u_x(0,t) &= 0, && t > 0, \\ u_x(1,t) &= A, && t > 0. \end{aligned}$$

Find the value of A such that the solution $u(x,t)$ has a steady state as t tends to infinity.

Question 3 [20 marks]

(a) Let λ be a positive number such that the boundary value problem

$$y'' + \lambda y = 0,$$

$$y(0) = 0, \quad y'(1) + y(1) = 0,$$

has a nontrivial solution.

(i) Prove that λ satisfies the equation

$$\sin\sqrt{\lambda} + \sqrt{\lambda}\cos\sqrt{\lambda} = 0.$$

(ii) Deduce that $\cos\sqrt{\lambda} \neq 0$.

(b) Let $u(x,t)$ be the solution to the boundary value problem

$$\frac{\partial^2 u(x,t)}{\partial t \partial x} = \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad -\infty < x < \infty,\ t > 0,$$

$$u(x,0) = \sqrt{\tfrac{\pi}{2}}\, e^{-|x|}, \qquad -\infty < x < \infty.$$

Evaluate $u(1,1)$.

(c) Let $u(x,t)$ be the solution to the boundary value problem

$$\begin{aligned} u_{xx}(x,t) &= u_{tt}(x,t), && -\infty < x < \infty,\ t > 0, \\ u(x,0) &= f(x), && -\infty < x < \infty, \\ u_t(x,0) &= g(x), && -\infty < x < \infty, \end{aligned}$$

where both f and g have Fourier transforms. Find the Fourier transform $\widehat{u}(\omega,t)$ of $u(x,t)$ in terms of $\widehat{f}(\omega)$ and $\widehat{g}(\omega)$.

Question 4 [20 marks]

(a) (i) Evaluate the integral

$$\int_\gamma \frac{dz}{z^2 - 4z + 1},$$

where γ is the positively oriented unit circle with centre at $(0,0)$.

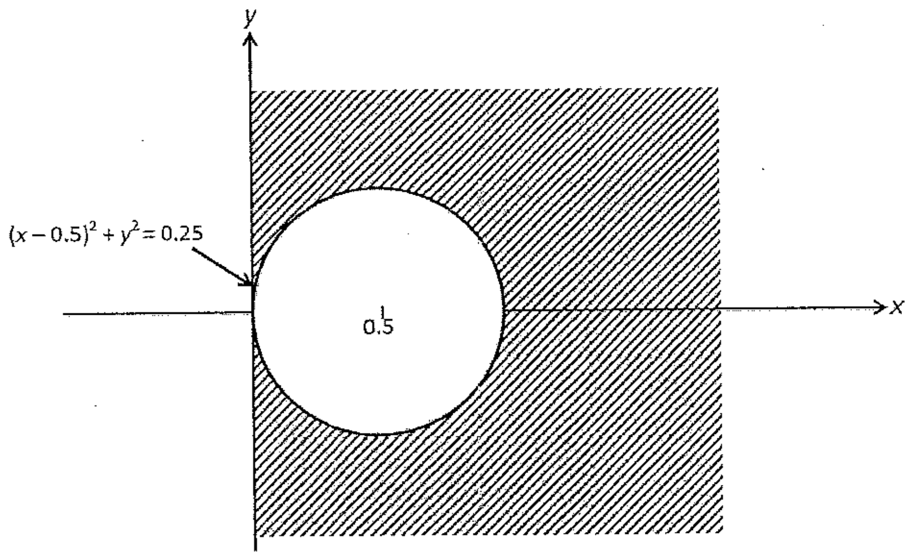
(ii) Evaluate the integral

$$\int_0^\pi \frac{d\theta}{2 - \cos\theta}.$$

(b) By using the mapping $w = \frac{1}{z}$, or otherwise, solve the Laplace equation

$$\begin{aligned} &\phi_{xx}(x,y) + \phi_{yy}(x,y) = 0 && \text{on} \quad D, \\ &\phi(x,y) = 50 && \text{on} \quad (x - \tfrac{1}{2})^2 + y^2 = \tfrac{1}{4}, \\ &\phi(0,y) = 10 && \text{on} \quad -\infty < y < \infty, \end{aligned}$$

where $D = \{(x,y) : x \geq 0, -\infty < y < \infty\} \setminus \{(x,y) : (x - \frac{1}{2})^2 + y^2 < \frac{1}{4}\}$, that is, D is the right half plane in which the disc $(x - \frac{1}{2})^2 + y^2 < \frac{1}{4}$ is cut out, as shown in the shaded region below.



END OF PAPER

TURN OVER FOR MATHEMATICAL TABLES

Boundary Value Problems $X''(x) + \lambda X(x) = 0$

Boundary conditions	$X(0)=0$ $X(L)=0$	$X'(0)=0$ $X'(L)=0$	$X(-L)=X(L)$ $X'(-L)=X'(L)$
Eigenvalues λ_n	$(\frac{n\pi}{L})^2$ $n=1,2,3,\dots$	$(\frac{n\pi}{L})^2$ $n=0,1,2,3,\dots$	$(\frac{n\pi}{L})^2$ $n=0,1,2,3,\dots$
Eigenfunctions	$\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$
Series	$f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}$ $+\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$
Coefficients	$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$	$A_0=\frac{1}{L}\int_0^L f(x)dx$ $A_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$	$A_0=\frac{1}{2L}\int_{-L}^L f(x)dx$ $A_n=\frac{1}{L}\int_{-L}^L f(x)\cos\frac{n\pi x}{L}dx$ $B_n=\frac{1}{L}\int_{-L}^L f(x)\sin\frac{n\pi x}{L}dx$

Boundary conditions	$X(0)=0$ $X'(L)=0$	$X'(0)=0$ $X(L)=0$
Eigenvalues λ_n	$\left[\frac{(2n-1)\pi}{2L}\right]^2$ $n=1,2,3,\dots$	$\left[\frac{(2n-1)\pi}{2L}\right]^2$ $n=1,2,3,\dots$
Eigenfunctions	$\sin\frac{(2n-1)\pi x}{2L}$	$\cos\frac{(2n-1)\pi x}{2L}$
Series	$f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{(2n-1)\pi x}{2L}$	$f(x)=\sum_{n=1}^{\infty} B_n \cos\frac{(2n-1)\pi x}{2L}$
Coefficients	$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}dx$	$B_n=\frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}dx$

Fourier transform $\mathcal{F}$ (with respect to x)

$$\mathcal{F}(e^{-a|x|}) = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+\omega^2} \qquad a>0$$

$$\mathcal{F}(\sqrt{\frac{2}{\pi}}\frac{a}{x^2+a^2}) = e^{-|\omega|a}, \qquad a>0$$

$$\mathcal{F}(e^{-ax^2}) = \sqrt{\frac{1}{2a}}\,e^{-\frac{\omega^2}{4a}}, \qquad a>0$$

$$\mathcal{F}^{-1}(\hat{f}(\omega)\hat{g}(\omega)) = (g*f)(x)$$

$$(g*f)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)g(x-s)\,ds$$

$$\mathcal{F}(\frac{\partial^n}{\partial t^n}u(x,t))(\omega) = \frac{d^n}{dt^n}\widehat{u}(\omega,t), \qquad n=1,2,3,\cdots$$

$$\mathcal{F}(\frac{\partial^n}{\partial x^n}u(x,t))(\omega) = (i\omega)^n\widehat{u}(\omega,t), \qquad n=1,2,3,\cdots$$

$$\mathcal{F}(\frac{\partial^2}{\partial t\partial x}u(x,t))(\omega) = i\omega\frac{d}{dt}\widehat{u}(\omega,t)$$

Fourier cosine transform $\mathcal{F}_c$ (with respect to x)

$$\mathcal{F}_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}(\frac{a}{a^2+\omega^2}), \qquad a>0$$

$$\mathcal{F}_c(e^{-ax^2}) = \sqrt{\frac{1}{2a}}\,e^{\frac{-\omega^2}{4a}}, \qquad a>0$$

$$\mathcal{F}_c(\frac{a}{a^2+x^2}) = \sqrt{\frac{\pi}{2}}\,e^{-a\omega}, \qquad a>0$$

Fourier sine transform $\mathcal{F}_s$ (with respect to x)

$$\mathcal{F}_s(e^{-ax}) = \sqrt{\frac{2}{\pi}}(\frac{\omega}{a^2+\omega^2}), \qquad a>0$$

$$\mathcal{F}_s(\frac{x}{a^2+x^2}) = \sqrt{\frac{\pi}{2}}\,e^{-a\omega}, \qquad a>0$$

Laplace transform $\mathcal{L}$ with respect to t

$\mathcal{L}(u(x,t))$ is denoted by $U(x,s)$

$\mathcal{L}(u_t(x,t)) = sU(x,s) - u(x,0)$

$\mathcal{L}(u_x(x,t)) = U_x(x,s)$

$\mathcal{L}(1) = \frac{1}{s}, \qquad s > 0$

$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \qquad s > 0,\ n = 1, 2, 3, \cdots$

$\mathcal{L}(\sin kt) = \frac{k}{s^2 + k^2}, \qquad s > 0$

$\mathcal{L}(\cos kt) = \frac{s}{s^2 + k^2}, \qquad s > 0$

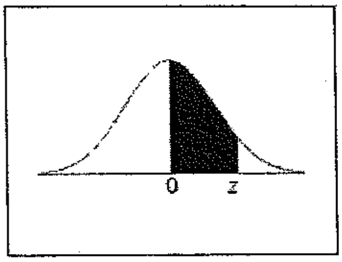
$\mathcal{L}(e^{at}) = \frac{1}{s - a}, \qquad s > a$

$\mathcal{L}(e^{at} f(t)) = (\mathcal{L}f)(s - a),$

$\mathcal{L}(H(t - a) f(t - a)) = e^{-as}(\mathcal{L}f)(s),$

where $H(r) = 0$ if $r < 0$, $H(r) = 1$ if $r \geq 0$

Standard Normal Distribution Table



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998