Matriculation Number:

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2011-2012

MA1506 MATHEMATICS II

April 2012 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
(a)								
(b)								

Question 1 (a) [5 marks]

Radium decomposes at a rate given by the differential equation

$$\frac{dQ}{dt} = -kQ$$

where Q is the amount of radium measured in units at time t measured in years and k is a positive constant. It is known that the half-life of Radium is equal to 1600 years. Starting at time $t = 0$ year with 10 units of radium, it is calculated that there are c units of radium at time $t = 2400$ years.

(i) Find the **exact** value of k. Give your answer in the form of a fraction in terms of $\ln 2$.

(ii) Find the value of c correct to one decimal place.

Answer 1(a)(i)	$\frac{\ln 2}{1600}$	Answer 1(a)(ii)	3.5

(Show your working below and on the next page.)

$$\frac{dQ}{dt} = -kQ, \quad Q(0) = 10$$

$$\Rightarrow Q = 10e^{-kt}$$

$$\frac{1}{2} = e^{-1600k}$$

$$\Rightarrow k = \underline{\underline{\frac{\ln 2}{1600}}}$$

$$c = 10e^{-\frac{\ln 2}{1600} \times 2400}$$

$$= 3.53\ldots$$

$$\approx \underline{\underline{3.5}}$$

Question 1 (b) [5 marks]

For obvious reasons, the dissecting room of a certain coroner is kept very cool at a constant temperature of 5^o C. One day the coroner's assistant arrived at the dissecting room at 10am and found that the coroner himself was murdered. He immediately took the coroner's body temperature and found it to be 23^o C. The body was then left in the room, so as not to disturb the crime scene, until the CSI team arrived at 12:00 noon. The CSI found that the victim's body temperature at 12:00 noon was a^o C. Based on the assumption that the coroner's body temperature was 37^o C when he was alive, the CSI calculated that the time of death was 6am. Find the value of a correct to one decimal place.
(Suggestion: Use hour as the unit of measurement of time and set time $t = 0$ hour at 6am.)

Answer 1(b)	18.5

(Show your working below and on the next page.)

$$\frac{dT}{dt} = k(T-5)$$

$$\Rightarrow T-5 = Ae^{kt}$$

$$T(0)=37 \Rightarrow A = 32$$

$$T = 5 + 32e^{kt}$$

$$T(4)=23 \Rightarrow 23 = 5 + 32e^{4k} \Rightarrow k = \frac{1}{4}(\ln 9 - \ln 16)$$

$$a = 5 + 32e^{\frac{1}{4}(\ln 9 - \ln 16)\times 6}$$

$$= 5 + 32 \times \left(\frac{3}{4}\right)^3$$

$$= \underline{\underline{18.5}}$$

Question 2 (a) [5 marks]

(i) Find the general solution of the differential equation

$$y'' + y' - 2y = 0.$$

(ii) Find the general solution of the differential equation

$$y'' - 2y' + 2y = 0.$$

Answer 2(a)(i)	$y = C_1 e^{-2x} + C_2 e^{x}$	Answer 2(a)(ii)	$y = C_1 e^{x}\cos x + C_2 e^{x}\sin x$

(Show your working below and on the next page.)

(i) $\lambda^2 + \lambda - 2 = 0$

$(\lambda + 2)(\lambda - 1) = 0$

$\lambda = -2, 1$

$$\underline{\underline{y = C_1 e^{-2x} + C_2 e^{x}}}$$

(ii) $\lambda^2 - 2\lambda + 2 = 0$

$\lambda = \dfrac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i$

$$\underline{\underline{y = C_1 e^{x}\cos x + C_2 e^{x}\sin x}}$$

Question 2 (b) [5 marks]

Consider the equation of motion of a particle at time t given by the second order ordinary differential equation

$$\ddot{x} = \frac{3}{2}x^2 - 1.$$

It is known that the equation of the phase plane curve which satisfies the initial conditions $x(0) = 1$, $y(0) = 1$, is of the form $y^2 = f(x)$, where $y = \dot{x}$ and $f(x)$ is a polynomial in x. Find the formula for $f(x)$.

Answer 2(b)	$x^3 - 2x + 2$ (or $y^2 = x^3 - 2x + 2$)

(Show your working below and on the next page.)

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx}\cdot\frac{dx}{dt} = \dot{x}\frac{d\dot{x}}{dx} = \frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right)$$

$$\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = \frac{3}{2}x^2 - 1$$

$$\frac{1}{2}\dot{x}^2 = \frac{1}{2}x^3 - x + C_1$$

$$y^2 = x^3 - 2x + C$$

$$x(0) = y(0) = 1 \Rightarrow 1 = 1 - 2 + C \Rightarrow C = 2$$

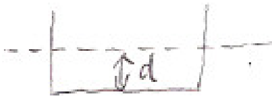
$$\underline{\underline{y^2 = x^3 - 2x + 2}}$$

Question 3 (a) [5 marks]

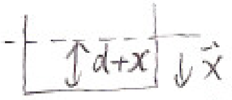
A cruise ship can be modeled as a solid object with perfectly vertical sides and a perfectly horizontal bottom, so all horizontal cross-sections have the same area, equal to 9000 m^2. Archimedes' principle states that the upward force exerted on a ship by the sea is equal to the weight of the water pushed aside by the ship. Let the density of seawater be 1027 kg/m^3, and let 9×10^6 kg be the mass of the ship. Assume that gravity and buoyancy are the only forces acting on the ship when the sea is calm and there is no friction when the ship bobs up and down. Now waves from a storm strike the ship and exert a vertical force $F\cos(\alpha t)$ on the ship, where F Newton is the amplitude of the wave force and α with a unit of 1/second is the wave frequency. Assume that the gravitational constant $g = 10$ m/s^2, what is the most dangerous value of α? Give your answer correct to one decimal place.

Answer 3(a)	3.2

(Show your working below and on the next page.)

stationary 

$1027 g \times 9000 \times d = 9 \times 10^6 \times g$

bobs up and down 

$$9 \times 10^6 \times \ddot{x} = 9 \times 10^6 \times g - 1027 g \times 9000 \times (d+x)$$
$$= -1027 g \times 9000 \times x$$
$$\ddot{x} = -10.27x \quad (\because g = 10\ m/s^2)$$

most dangerous α occurs at resonance

i.e. $\alpha = \sqrt{10.27} \approx \underline{\underline{3.2}}$

Question 3 (b) [5 marks]

(i) Let $x(t)$ be the solution of the initial value problem

$$\frac{dx}{dt} = x^2 - 30x + 200, \text{ and } x(0) = 18.$$

Find the **exact** value of $\lim_{t\to\infty} x(t)$.

(ii) Let $x(t)$ be the solution of the initial value problem

$$\frac{dx}{dt} = x^2 - 6x + 9, \text{ and } x(0) = k.$$

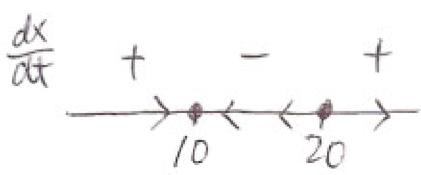
Find the largest integer k which satisfies $\lim_{t\to\infty} x(t)$ exists and is finite.

Answer 3(b)(i)	10	**Answer 3(b)(ii)**	3

(Show your working below and on the next page.)

(i) $\frac{dx}{dt} = x^2 - 30x + 200$

$= (x-10)(x-20)$

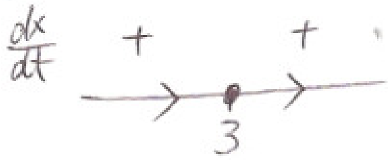


$x(0) = 18$

$\Rightarrow x(t) \to \underline{\underline{10}}$

(ii) $\frac{dx}{dt} = x^2 - 6x + 9$

$= (x-3)^2$



$x(0) \leq 3 \Rightarrow x(t) \to 3$

$x(0) > 3 \Rightarrow x(t) \to \infty$

$\therefore k = \underline{\underline{3}}$

Question 4 (a) [5 marks]

For many years the deer in Minnesota were protected by law and eventually they settled down to a logistic equilibrium population of 200000 with a birth rate per capita of 10% per year. Then the state government decided to lift the hunting ban and allowed the hunters to shoot E deer per year. What is the value of E if the state government wants the deer population to be able to bounce back from any natural disaster that pushes the population down by 20%? Give your answer correct to the nearest integer.

Answer 4(a)	4938

(Show your working below and on the next page.)

$$\frac{dN}{dt} = BN - SN^2 - E, \quad \frac{B}{S} = 200000, \ B = 0.1$$

$$= -S\left(N^2 - \frac{B}{S}N + \frac{E}{S}\right)$$

$$\beta_1 = \frac{B/S - \sqrt{B^2/S^2 - \frac{4E}{S}}}{2}, \quad \beta_2 = \frac{B/S + \sqrt{B^2/S^2 - \frac{4E}{S}}}{2}$$

$$\beta_1 = 0.8\beta_2 \Rightarrow \frac{B}{S} - \sqrt{\frac{B^2}{S^2} - \frac{4E}{S}} = \frac{0.8B}{S} + 0.8\sqrt{\frac{B^2}{S^2} - \frac{4E}{S}}$$

$$\Rightarrow (0.2)\frac{B}{S} = (1.8)\sqrt{\frac{B^2}{S^2} - \frac{4E}{S}}$$

$$\Rightarrow \frac{B^2}{S^2} = 81\frac{B^2}{S^2} - \frac{324E}{S}$$

$$\Rightarrow E = \frac{80 \times \frac{B}{S} \times B}{324} = \frac{400000}{81}$$

$$= 4938.27\cdots \approx \underline{\underline{4938}}$$

Question 4 (b) [5 marks]

Find the solution of the initial value problem

$$y'' - y' = \delta(t-3), \quad y(0) = y'(0) = 0,$$

where δ is the Dirac delta function.

Answer 4(b)	$y = (e^{t-3} - 1)\,u(t-3)$

(Show your working below and on the next page.)

Take Laplace transform and set $Y = L(y)$,

$$s^2 Y - s y(0) - y'(0) - sY + y(0) = e^{-3s}$$

$$Y = \frac{e^{-3s}}{s^2 - s}$$

$$= \left(\frac{1}{s-1} - \frac{1}{s}\right) e^{-3s}$$

$$\because L^{-1}\left(\frac{1}{s-1} - \frac{1}{s}\right) = e^t - 1$$

$$\therefore \underline{\underline{y = (e^{t-3} - 1)\,u(t-3)}}$$

Question 5 (a) [5 marks]

Let $M = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$. Find the **exact** value of M^{1506}.

(Hint: you may want to use the fact that $\sin 60^o = \frac{\sqrt{3}}{2}$ and $\cos 60^o = \frac{1}{2}$.)

Answer 5(a)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(Show your working below and on the next page.)

$$M = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{pmatrix}$$

$= \text{rotation by } 60^\circ$

$\therefore M^{1506} = \text{rotation by } 1506 \times 60^\circ$

$= \text{rotation by } 251 \times 360^\circ$

$$= \underline{\underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}}.$$

Question 5 (b) [5 marks]

A small village is divided into two communities. Group A grows rice but likes to eat chicken rice. For group A to produce one dollar's worth of rice, it costs them 40 cents' worth of rice and ten cents' worth of chicken. Group B raises chicken and also eats chicken rice. For group B to produce one dollar's worth of chicken, it costs them 30 cents' worth of chicken and 20 cents' worth of rice. The village wants to sell \$4000 of rice and \$4000 of chicken each year to the outside world.

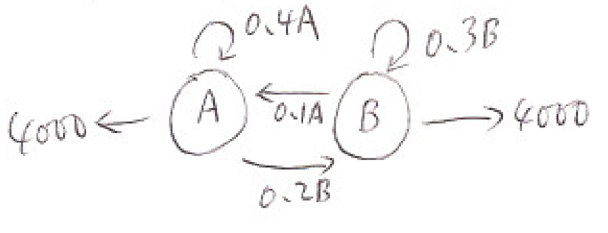
(i) Find, in terms of dollar value, how much rice should group A produce per year.

(ii) Find, in terms of dollar value, how much chicken should group B produce per year.

Give **exact** values for your answers.

Answer 5(b)(i)	9000	**Answer 5(b)(ii)**	7000

(Show your working below and on the next page.)



$$A = 0.4A + 0.2B + 4000$$
$$B = 0.1A + 0.3B + 4000$$

$$\begin{pmatrix} 0.6 & -0.2 \\ -0.1 & 0.7 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 4000 \\ 4000 \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{0.4}\begin{pmatrix} 0.7 & 0.2 \\ 0.1 & 0.6 \end{pmatrix}\begin{pmatrix} 4000 \\ 4000 \end{pmatrix} = \begin{pmatrix} 9000 \\ 7000 \end{pmatrix}$$

Question 6 (a) [5 marks]

Each year 10% of the residents in Ang Mo Kio leave their area and move to some other parts of Singapore, while 20% of the Singapore population outside of Ang Mo Kio move into the Ang Mo Kio area. Assume that the Singapore population stays constant each year, and denote the number of people in Singapore living inside and outside Ang Mo Kio at the end of the year $2000 + n$ by a_n and b_n respectively. Find the 2×2 matrix M such that

$$\begin{pmatrix} a_{n+2} \\ b_{n+2} \end{pmatrix} = M \begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

Give your answer correct to two decimal places for each entry of M.

Answer 6(a)	$M = \begin{pmatrix} 0.83 & 0.34 \\ 0.17 & 0.66 \end{pmatrix}$

(Show your working below and on the next page.)

$$a_{n+1} = a_n - 0.1a_n + 0.2b_n$$

$$b_{n+1} = b_n - 0.2b_n + 0.1a_n$$

$$\Rightarrow \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

$$\begin{pmatrix} a_{n+2} \\ b_{n+2} \end{pmatrix} = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 0.83 & 0.34 \\ 0.17 & 0.66 \end{pmatrix}}} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

Question 6 (b) [5 marks]

Suppose the weather in a certain town is governed by a Markov Chain described by a two by two matrix M. The weather can be either rainy or sunny. Suppose that we know that the determinant of M is equal to $\frac{1}{4}$, and that if it is sunny today, then the probability that it will be sunny tomorrow is 0.5.

(i) If it is sunny today, what is the probability that it will rain tomorrow?

(ii) If it is rainy today, what is the probability that it will rain tomorrow?

Answer 6(b)(i)	0.5	Answer 6(b)(ii)	0.75

(Show your working below and on the next page.)

$$\begin{array}{c|cc} & R & S \\ \hline R & & \\ S & & \end{array} \quad \begin{pmatrix} x & 0.5 \\ 1-x & 0.5 \end{pmatrix} \Rightarrow \begin{vmatrix} x & 0.5 \\ 1-x & 0.5 \end{vmatrix} = \frac{1}{4}$$

$$x - 1 + x = \frac{1}{2}$$

$$2x = 1.5$$

$$x = \underline{\underline{0.75}}$$

Question 7 (a) [5 marks]

(i) Classify the following system of differential equations (that is, state whether it represents a nodal source, nodal sink, saddle, spiral source, spiral sink, or centre).

$$\begin{cases} \frac{dx}{dt} = 3x - y \\ \frac{dy}{dt} = -x + 3y \end{cases}.$$

(ii) It is known that the following system of differential equations represents a spiral source. Determine whether the direction of flow is clockwise or anti-clockwise.

$$\begin{cases} \frac{dx}{dt} = 4x - 3y \\ \frac{dy}{dt} = 5x - 3y \end{cases}.$$

Answer 7(a)(i)	Nodal Source	Answer 7(a)(ii)	anti-clockwise

(Show your working below and on the next page.)

(i) $\det = 8$, $\text{Tr} = 6$

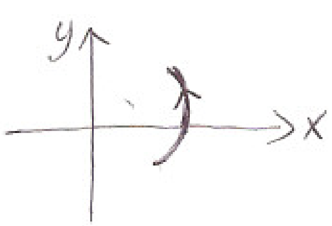
$\text{Tr}^2 - 4\det = 36 - 32 = +ve$

$\Rightarrow$ Nodal Source.

(ii) Take a point with $x > 0$, $y = 0$

$\Rightarrow \frac{dy}{dt} = 5x - 3y = +ve$

$\Rightarrow y \uparrow$



$\Rightarrow$ anti-clockwise

Question 7 (b) [5 marks]

Let $x(t)$ and $y(t)$ be a solution of the following system of differential equations:

$$\begin{cases} \frac{dx}{dt} = x + 5y \\ \frac{dy}{dt} = x - 3y \end{cases}.$$

It is known that $\lim_{t\to\infty} x(t) = \infty$.

Find the **exact** value of the following limit:

$$\lim_{t\to\infty} \frac{y(t)}{x(t)}.$$

(Hint: you may want to use the fact that the eigenvalues of the matrix $\begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}$ are -4 and 2.)

Answer 7(b)	$\frac{1}{5}$

(Show your working and drawing below and on the next page.)

$\lambda = -4 \Rightarrow 5x + 5y = 0 \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector

$\lambda = 2 \Rightarrow -x + 5y = 0 \Rightarrow \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ is an eigenvector

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$\lim_{t\to\infty} x(t) = \infty \Rightarrow C_2 \neq 0$

$$\lim_{t\to\infty} \frac{y(t)}{x(t)} = \lim_{t\to\infty} \frac{-C_1 e^{-4t} + C_2 e^{2t}}{C_1 e^{-4t} + 5C_2 e^{2t}}$$

$$= \lim_{t\to\infty} \frac{-C_1 e^{-6t} + C_2}{C_1 e^{-6t} + 5C_2} = \frac{C_2}{5C_2} = \underline{\underline{\frac{1}{5}}}$$

Question 8 (a) [5 marks]

The battle of Trafalgar in 1805 began with 50 British ships and 100 French ships. With the British ships having twice the firing power as the French ships, we càn model a typical sea battle by the following system of differential equations:

$$\begin{cases} \frac{dx}{dt} = -y \\ \frac{dy}{dt} = -2x \end{cases}$$

where x and y denote the number of British and French ships respectively at any time t. The British commander at Trafalgar, Admiral Lord Nelson, knew that his side would be doomed in a direct confrontation with the French side as they were vastly out-numbered by the French ships. In order to save some ships from his side so as to fight another day, Lord Nelson devised the following battle plan: He divided his ships into a large group of 36 ships and a small group of 14 ships and he managed to arrange the battle into two separate and independent sub-battles so that the larger British group engaged 30 French ships while the smaller British group engaged 70 French ships. It is known that all ships in the smaller British group were destroyed and all ships in the smaller French group were also destroyed. Find out how many British ships survived the battle. Give your answer correct to the nearest integer.
(Hint: you may want to use the fact that the eigenvalues of the matrix $\begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}$ are $-\sqrt{2}$ and $\sqrt{2}$.)

Answer 8(a)	29

(Show your working below and on the next page.)

$\lambda = -\sqrt{2} \Rightarrow \sqrt{2}x - y = 0 \Rightarrow \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$ is an eigenvector

$\lambda = \sqrt{2} \Rightarrow -\sqrt{2}x - y = 0 \Rightarrow \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$ is an eigenvector

$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{-\sqrt{2}t} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + C_2 e^{\sqrt{2}t} \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$

$t = 0 \Rightarrow \begin{pmatrix} 36 \\ 30 \end{pmatrix} = \begin{pmatrix} C_1 + C_2 \\ \sqrt{2}C_1 - \sqrt{2}C_2 \end{pmatrix} \Rightarrow C_1 = \frac{1}{2}\left(36 + \frac{30}{\sqrt{2}}\right)$

$C_2 = \frac{1}{2}\left(36 - \frac{30}{\sqrt{2}}\right)$

(More working space for Question 8(a))

$$y=0 \Rightarrow \sqrt{2}C_1e^{-\sqrt{2}t}-\sqrt{2}C_2e^{\sqrt{2}t}=0$$

$$\Rightarrow e^{2\sqrt{2}t}=\frac{C_1}{C_2}$$

$$\Rightarrow t=\frac{1}{2\sqrt{2}}(\ln C_1-\ln C_2)$$

$$\approx 0.478$$

$$\Rightarrow x=C_1e^{-\sqrt{2}t}+C_2e^{\sqrt{2}t}$$

$$\approx \frac{1}{2}\left(36+\frac{30}{\sqrt{2}}\right)e^{-\sqrt{2}\times 0.478}+\frac{1}{2}\left(36-\frac{30}{\sqrt{2}}\right)e^{\sqrt{2}\times 0.478}$$

$$\approx 29.086\ldots$$

$$\approx \underline{\underline{29}}$$

Question 8 (b) [5 marks]

Using the method of separation of variables, find the solution $u(x,y)$ of the partial differential equation

$$u_y + \frac{1}{2}u_x = 0,$$

which satisfies $u(x,0) = e^{2x}$.

Answer 8(b)	$u = e^{2x-y}$

(Show your working below and on the next page.)

Let $u = XY$

$\Rightarrow XY' + \frac{1}{2}X'Y = 0$

$\Rightarrow 2XY' = -X'Y$

$\Rightarrow \frac{2Y'}{Y} = -\frac{X'}{X} = k$

$\Rightarrow Y = Ae^{\frac{1}{2}ky}$ and $X = Be^{-kx}$

$\therefore u = XY = Ce^{\frac{1}{2}ky - kx}$

$u(x,0) = e^{2x} \Rightarrow e^{2x} = Ce^{-kx}$

$\Rightarrow C = 1,\ k = -2$

$\therefore \underline{\underline{u = e^{2x-y}}}$