Matriculation Number:

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2010-2011

MA1506 MATHEMATICS II

April 2011 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
(a)								
(b)								

Question 1 (a) [5 marks]

The intensity, I, of light at a depth of x meters below the surface of the Singapore River satisfies the differential equation

$$\frac{dI}{dx} = (-0.9)\, I.$$

Find the distance below the surface, correct to two decimal places in meters, at which the intensity of light is equal to half the intensity at the surface.

Answer 1(a)	0.77

(Show your working below and on the next page.)

$$\frac{dI}{I} = -0.9\,dx$$

$$I = Ae^{-0.9x}$$

$$\tfrac{1}{2}A = Ae^{-0.9x}$$

$$x = \frac{-\ln 2}{-0.9} \approx \underline{\underline{0.77}}$$

Question 1 (b) [5 marks]

Romeo was standing directly below Juliet's balcony. He threw a stone vertically upwards at a velocity of $20m/s$. To his horror, T seconds after he threw the stone, Juliet stuck her head out of the balcony and the stone hit her on the face on its way up. Luckily for Juliet (and so also for Romeo), at the moment of impact the velocity of the stone was zero. Find the value of T correct to two decimal places, based on the following assumptions: the stone's mass is $0.3kg$, the gravitational constant g equals to $9.8m/s^2$ and the value of the air resistance at any time equals to $0.3v$ Newtons where v is the value of the velocity of the stone at that time measured in m/s.

Answer 1(b)	1.11

(Show your working below and on the next page.)

$$0.3\frac{dv}{dt} = -0.3g - 0.3V$$

$$\frac{dV}{dt} = -(g+V)$$

$$\frac{dV}{g+V} = -dt$$

$$g+V = Ae^{-t}$$

$$t=0,\ V=20 \Rightarrow 9.8+20 = A$$

$$\therefore\ 9.8+V = 29.8e^{-t}$$

$$t=T,\ V=0 \Rightarrow 9.8 = 29.8e^{-T}$$

$$T = \ln(29.8) - \ln(9.8)$$

$$= 1.112126\cdots$$

$$\approx \underline{\underline{1.11}}$$

Question 2 (a) [5 marks]

(i) Find the general solution of the differential equation

$$y'' - y' - 2y = 0.$$

(ii) Find the general solution of the differential equation

$$y'' - 2y' + y = 0.$$

Answer 2(a)(i)	$y = C_1 e^{-x} + C_2 e^{2x}$	Answer 2(a)(ii)	$y = C_1 e^{x} + C_2 x e^{x}$

(Show your working below and on the next page.)

(i) $\lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1) = 0$

$\lambda = -1, 2$

$$y = C_1 e^{-x} + C_2 e^{2x}$$

(ii) $\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda = 1$ double root

$$y = C_1 e^{x} + C_2 x e^{x}$$

Question 2 (b) [5 marks]

The growth of the population in Singapore follows a logistic model with a birth rate per capita of 1.16% per year. Currently the population in Singapore has reached its logistic equilibrium of four millions. The Singapore government has decided to allow a constant number, E, of foreigners to settle down in Singapore each year so as to reach a target equilibrium population of six millions. Find the **exact** value of E.

Answer 2(b)	0.0348 million = 34800

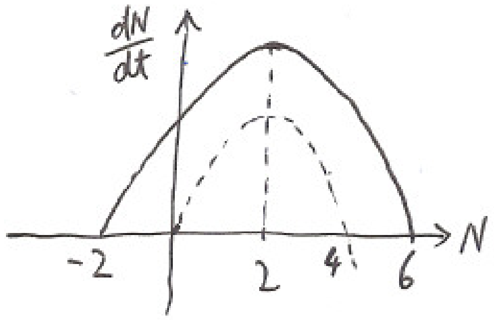
(Show your working below and on the next page.)

Let N = population in millions

$$\frac{dN}{dt} = BN - SN^2 + E$$

$$B = 1.16\% = 0.0116, \quad S = \frac{B}{4} = 0.0029$$

$$\therefore \frac{dN}{dt} = -0.0029N^2 + BN + E \text{ ------ } (1)$$



$-2, 6$ are roots of (1)

Product of root $= (-2)(6) = \dfrac{E}{-0.0029}$

$$\therefore E = 0.0348 \text{ million} = \underline{\underline{34800}}$$

Question 3 (a) [5 marks]

A circus clown tries to balance a pendulum at its unstable equilibrium point. At the start of the performance, the pendulum is slightly away from its unstable equilibrium and it is initially at rest. The length of the pendulum is $L = 39.2$ cm, and the clown's skill is such that he can stop the pendulum from falling provided that the angular deviation from the unstable equilibrium position does not exceed three times the initial value. Estimate the speed of his reflexes. Give your answer in seconds correct to two decimal places. Assumption: the gravitational constant g equals to $9.8m/s^2$.
(Hint: You may want to use the fact that the angular deviation from the unstable equilibrium position, ϕ, at any time t, satisfies the differential equation $\ddot{\phi} = \frac{g}{L}\phi$.)

Answer 3(a)	0.35

(Show your working below and on the next page.)

$$\ddot{\phi} = \frac{g}{L}\phi$$

$$\lambda^2 = \frac{g}{L} \Rightarrow \lambda = \pm\sqrt{\frac{g}{L}} = \pm\sqrt{\frac{9.8}{0.392}} = \pm 5$$

$$\therefore \phi = C_1 e^{5t} + C_2 e^{-5t}$$

Let $\phi(0) = \varepsilon$, $\phi'(0) = 0 \Rightarrow C_1 = C_2 = \frac{\varepsilon}{2} \Rightarrow \phi = \frac{\varepsilon}{2}(e^{5t} + e^{-5t})$

$$\therefore \phi = \varepsilon \cosh 5t$$

when $\phi = 3\varepsilon$, we have

$$3\varepsilon = \varepsilon \cosh 5t$$

$$\therefore t = \frac{1}{5}\cosh^{-1}(3) = 0.352\cdots$$

$$\approx \underline{\underline{0.35}}$$

Question 3 (b) [5 marks]

Let $x(t)$ be the solution of the initial value problem

$$\frac{dx}{dt} = x \sinh\left(100 - 30x + 2x^2\right), \text{ and } x(0) = k.$$

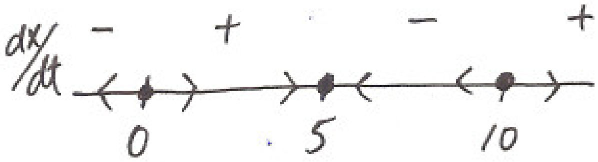
(i) If $k = 9$, find the **exact** value of $\lim_{t\to\infty} x(t)$.

(ii) If $k = 10$, find the **exact** value of $\lim_{t\to\infty} x(t)$.

Answer 3(b)(i)	5	**Answer 3(b)(ii)**	10

(Show your working below and on the next page.)

$$\frac{dx}{dt} = x \sinh\{(20-2x)(5-x)\}$$



(i) $x(0) = 9 \Rightarrow x \to \underline{\underline{5}}$

(ii) $x(0) = 10 \Rightarrow x \equiv 10 \Rightarrow x \to \underline{\underline{10}}$

Question 4 (a) [5 marks]

A horizontal spring system has one end of the spring fixed to a vertical wall and a weight attached at the other end. The weight lies on a perfectly smooth horizontal floor. The system is initially at rest and it is then set into a small oscillation about its equilibrium position. At time $t = \frac{\pi}{4}$ seconds the weight is suddenly struck with a hard blow in a direction parallel to the spring. If x measures the displacement of the weight from the equilibrium position at any time t, then it is found that the system can be modelled by the initial value problem

$$x'' + x = A\delta\left(t - \frac{\pi}{4}\right), \ x(0) = 1 \text{ and } x'(0) = -1,$$

where δ is the Dirac delta function. Find the **exact** value of A if it is known that $x(t) = 0$ for all $t > \frac{\pi}{4}$.

Answer 4(a)	$\sqrt{2}$

(Show your working below and on the next page.)

$$L(x''+x) = L\left(A\delta\left(t-\frac{\pi}{4}\right)\right)$$

$$\Rightarrow s^2X - s\,x(0) - x'(0) + X = Ae^{-\frac{\pi}{4}s}, \quad \text{where } X = L(x)$$

$$(s^2+1)X = s - 1 + Ae^{-\frac{\pi}{4}s}$$

$$X = \frac{s-1+Ae^{-\frac{\pi}{4}s}}{s^2+1}$$

$$= \frac{s}{s^2+1} - \frac{1}{s^2+1} + \frac{A}{s^2+1}e^{-\frac{\pi}{4}s}$$

$$\therefore x(t) = \cos t - \sin t + A\sin\left(t-\frac{\pi}{4}\right)u\left(t-\frac{\pi}{4}\right)$$

$$t > \frac{\pi}{4}, \ x(t) = 0 \Rightarrow 0 = \cos t - \sin t + A\sin t\cos\frac{\pi}{4} - A\cos t\sin\frac{\pi}{4}$$

$$= \left(1 - \frac{A}{\sqrt{2}}\right)(\cos t - \sin t) \quad \forall t > \frac{\pi}{4}$$

$$\therefore A = \underline{\underline{\sqrt{2}}}$$

Question 4 (b) [5 marks]

An RLC circuit consists of an inductor of inductance 1 henry, a resistor of resistance 2 ohms, a capacitor of capacitance 1 farad, a switch and a battery of voltage 3 volts connected in series. At time $t = 0$, the switch is in the open position and there is no electric current in the circuit. The switch is then immediately turned on and stays on until at time $t = 1$ when it is turned off. It is known that the electric current I at any time t satisfies the initial value problem

$$\frac{dI}{dt} + 2I + \int_0^t I(u)\,du = 3\left[1 - u(t-1)\right], \quad I(0) = 0,$$

where u is the unit step function. Find an **exact** expression for the current I at any time t.
(Hint: You may want to use the formula $L\left(\int_0^t I(u)\,du\right) = \frac{1}{s}L(I(t))$, where L denotes the Laplace transform.)

Answer 4(b)	$I = 3te^{-t} - 3(t-1)e^{-(t-1)}u(t-1)$

(Show your working below and on the next page.)

$$sL(I) - I(0) + 2L(I) + \frac{1}{s}L(I) = \frac{3}{s} - \frac{3e^{-s}}{s}$$

$$(s^2+2s+1)L(I) = 3 - 3e^{-s}$$

$$L(I) = \frac{3-3e^{-s}}{s^2+2s+1}$$

$$= \frac{3}{(s+1)^2} - \frac{3}{(s+1)^2}e^{-s}$$

$$\underline{\underline{I = 3te^{-t} - 3(t-1)e^{-(t-1)}u(t-1)}}$$

Question 5 (a) [5 marks]

Suppose you take a piece of rubber in two dimensions and shear it parallel to the x axis by 30 degrees, and then shear it parallel to the x axis again by 30 degrees. It is known that the result is the same if you just shear it parallel to the x axis once by an angle θ. Find the value of θ. Give your answer in **degrees** correct to the nearest degree.

Answer 5(a)	49°

(Show your working below and on the next page.)

Shear by $30^\circ \sim \begin{pmatrix} 1 & \tan 30^\circ \\ 0 & 1 \end{pmatrix}$

Shear by 30° twice $\sim \begin{pmatrix} 1 & \tan 30^\circ \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2\tan 30^\circ \\ 0 & 1 \end{pmatrix}$

Shear by $\theta \sim \begin{pmatrix} 1 & \tan\theta \\ 0 & 1 \end{pmatrix}$

$\therefore \tan\theta = 2\tan 30^\circ$

$\therefore \theta = \tan^{-1}\frac{2}{\sqrt{3}} \simeq 0.857$ radians

$= \frac{360}{2\pi} \times \tan^{-1}\left(\frac{2}{\sqrt{3}}\right) \simeq \underline{\underline{49^\circ}}$

Question 5 (b) [5 marks]

Let r_n and w_n denote the number of rabbits and weasels respectively after n years. Suppose that

$$\begin{pmatrix} r_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} r_n \\ w_n \end{pmatrix}$$

and $r_0 = 100$, $w_0 = 10$. Find the **exact** value of $\lim_{n\to\infty} \frac{r_n}{w_n}$.

(Hint: You may use the fact that the eigenvalues of $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$ are 2 and 3. The corresponding eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ respectively.)

Answer **5(b)**	2

(Show your working below and on the next page.)

$$P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \Rightarrow \det P = -1 \Rightarrow P^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} r_n \\ w_n \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} r_0 \\ w_0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 100 \\ 10 \end{pmatrix} = \begin{pmatrix} 180\times 3^n - 80\times 2^n \\ 90\times 3^n - 80\times 2^n \end{pmatrix}$$

$$\lim_{n\to\infty} \frac{r_n}{w_n} = \lim_{n\to\infty} \frac{180 - 80\times\left(\frac{2}{3}\right)^n}{90 - 80\times\left(\frac{2}{3}\right)^n} = \underline{\underline{2}}$$

Question 6 (a) [5 marks]

The billionaire engineer, Ms. Tan Ah Lian, credits her enormous success to the fact that she never talked and she always paid attention in her classes at NUS. One day she learned Markov Chains in MA1506. Using that knowledge, she classifies a day as a Down day if the Straits Times Index drops more than 10 points, an Ordinary day if it stays within a 10 points range from the previous closing, and a Good day if it goes up by more than 10 points. She observes that, if today is a Down day, then the probability that tomorrow will be a Down day and the probability that tomorrow will be an Ordinary day are 0.5 and 0.3 respectively. If today is an Ordinary day, then the probability that tomorrow will be Down day and the probability that tomorrow will be an Ordinary day are 0.3 and 0.5 respectively. If today is a Good day, then the probability that tomorrow will be a Down day and the probability that tomorrow will be an Ordinary day are 0.2 and 0.5 respectively. Based on her observations, she constructed a Markov matrix which she called a DOG matrix for the events: D (down day), O (ordinary day) and G (good day). She used her DOG matrix to play the stock market and made her first million dollars. The rest, as they say, is history. Use Tan Ah Lian's DOG matrix to answer the following two questions. Give **exact** values for your answers.

(i) If today, a Tuesday, is an Ordinary day, find the probability that two days later, a Thursday, is a Down day.

(ii) If today, a Tuesday, is a Good day, find the probability that three days later, a Friday, is also a Good day.

Answer 6(a)(i)	0.34	Answer 6(a)(ii)	0.223

(Show your working below and on the next page.)

$$\begin{array}{c c} & \begin{array}{ccc} D & O & G \end{array} \\ \begin{array}{c} D \\ O \\ G \end{array} & \underbrace{\begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.3 & 0.5 & 0.5 \\ 0.2 & 0.2 & 0.3 \end{pmatrix}}_{A} \end{array}$$

$$A^2 = \begin{pmatrix} 0.38 & 0.34 & 0.31 \\ 0.4 & 0.44 & 0.46 \\ 0.22 & 0.22 & 0.23 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0.354 & 0.346 & 0.339 \\ 0.424 & 0.432 & 0.438 \\ 0.222 & 0.222 & 0.223 \end{pmatrix}$$

(i) $\underline{\underline{0.34}}$

(ii) $\underline{\underline{0.223}}$

Question 6 (b) [5 marks]

Let A be a 4×4 matrix with entries from 1 to 16 such that each number is used exactly once and that A is a 4×4 magic square, that is, all rows, all columns, and the two diagonals have the same sum. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ denote the four eigenvalues of A. Find the **exact** value of $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$.

Answer 6(b)	34

(Show your working below and on the next page.)

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \mathrm{Tr}\,A$$

$$= \text{sum of diagonal}$$

$$= \text{Sum of one row}$$

$$= \frac{1}{4}(\text{sum of all 4 rows})$$

$$= \frac{1}{4}(1+2+\cdots+16)$$

$$= \frac{1}{4} \times \frac{17 \times 16}{2}$$

$$= \underline{\underline{34}}$$

Question 7 (a) [5 marks]

(i) Solve the following system of differential equations:

$$\begin{cases} \frac{dx}{dt} = 7x - 12y \\ \frac{dy}{dt} = 4x - 7y \end{cases}$$

with $x(0) = 1$ and $y(0) = 1$.

(Hint: You may use the fact that the eigenvalues of $\begin{pmatrix} 7 & -12 \\ 4 & -7 \end{pmatrix}$ are 1 and -1. The corresponding eigenvectors are $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ respectively.)

(ii) Classify the following system of differential equations (that is, say whether it represents a nodal source, nodal sink, saddle, spiral source, spiral sink, or centre.)

$$\begin{cases} \frac{dx}{dt} = x - 4y \\ \frac{dy}{dt} = 3x - 5y \end{cases}$$

Answer 7(a)(i)	$x = -2e^{t} + 3e^{-t}$ $y = -e^{t} + 2e^{-t}$	Answer 7(a)(ii)	spiral sink

(Show your working below and on the next page.)

(i) $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2c_1 e^{t} + 3c_2 e^{-t} \\ c_1 e^{t} + 2c_2 e^{-t} \end{pmatrix}$

$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} 2c_1 + 3c_2 = 1 \\ c_1 + 2c_2 = 1 \end{cases} \Rightarrow c_1 = -1, \ c_2 = 1$

$\underline{\underline{x = -2e^{t} + 3e^{-t}, \quad y = -e^{t} + 2e^{-t}}}$

(ii) $\det = 7$, $Tr = -4$, $\det > \frac{1}{4}(Tr)^2$

$\therefore$ $\underline{\underline{\text{spiral sink}}}$

Question 7 (b) [5 marks]

Two kinds of bacteria A and B are fighting with each other. They can reproduce themselves at a certain birth rate per capita and they are also killing each other at a rate proportional to the other kind at any instant. Suppose that we model this situation using the system of ordinary differential equations

$$\begin{cases} \frac{dx}{dt} = 3x - y \\ \frac{dy}{dt} = -2x + 2y \end{cases}$$

where x and y denote the number of bacteria A and bacteria B respectively at any time t. At time $t = 0$, there are 100 bacteria A fighting against k bacteria B. What is the maximum positive integer N that k **must be less than** (i.e. $k < N$) in order that all bacteria B will eventually be killed.

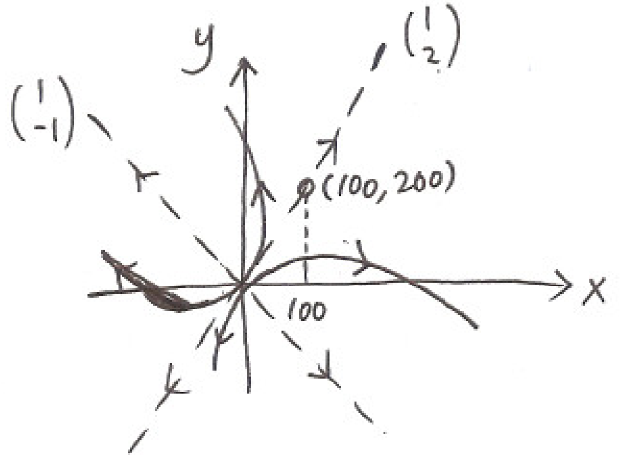
Answer 7(b)	200

(Show your working and drawing below and on the next page.)

$$\begin{vmatrix} 3-\lambda & -1 \\ -2 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-1)(\lambda-4)=0 \Rightarrow \lambda = 1, 4$$

$\lambda = 1 \Rightarrow 2x - y = 0 \Rightarrow \binom{1}{2}$ is an eigenvector

$\lambda = 4 \Rightarrow -x - y = 0 \Rightarrow \binom{1}{-1}$ is an eigenvector



$\underline{\underline{N = 200}}$

Question 8 (a) [5 marks]

Find the solution $u(x, y)$ of the partial differential equation

$$u_{xx} - 2u_x - 3u = 0$$

which satisfies $u(0, y) = 2y$ and $u_x(0, y) = 2y - 4\sin y$.

Answer 8(a)	$u = (y + \sin y)e^{-x} + (y - \sin y)e^{3x}$

(Show your working below and on the next page.)

Treat u as a function of x only

$$u'' - 2u' - 3u = 0$$

$$\lambda^2 - 2\lambda - 3 = 0 \Rightarrow \lambda = -1, 3$$

$$u = A(y)e^{-x} + B(y)e^{3x}$$

$$u(0,y) = 2y \Rightarrow A + B = 2y$$

$$u_x(0,y) = 2y - 4\sin y \Rightarrow -A + 3B = 2y - 4\sin y$$

$$\therefore\ 4B = 4y - 4\sin y$$

$$B = y - \sin y$$

$$A = 2y - B = y + \sin y$$

$$\underline{\underline{u = (y + \sin y)e^{-x} + (y - \sin y)e^{3x}}}$$

Question 8 (b) [5 marks]

Using the method of separation of variables, find the solution $u(x,y)$ of the partial differential equation

$$xu_x = u_y, \quad x > 0,$$

which satisfies $u(1,0) = 9$ and $u(2,0) = 36$.

Answer 8(b)	$u = 9x^2 e^{2y}$

(Show your working below and on the next page.)

Let $u = X(x)Y(y)$

$xX'Y = XY'$

$$\frac{xX'}{X} = \frac{Y'}{Y} = k$$

$$\frac{X'}{X} = \frac{k}{x} \Rightarrow \ln|X| = \ln x^k + C_1 \Rightarrow X = Ax^k$$

$$\frac{Y'}{Y} = k \Rightarrow \ln|Y| = ky + C_2 \Rightarrow Y = Be^{ky}$$

$\therefore u = Cx^k e^{ky}$

$u(1,0) = 9 \Rightarrow C = 9$

$u(2,0) = 36 \Rightarrow 9(2)^k = 36 \Rightarrow k = 2$

$$\underline{\underline{u = 9x^2 e^{2y}}}$$