Matriculation Number:

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2009-2010

MA1506 MATHEMATICS II

April 2010 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
Marks								

Question 1 (a) [5 marks]

Let $y(x)$ be the solution of the initial value problem

$$y\frac{dy}{dx} = x^2, \quad y > 0, \quad \text{and} \quad y(0) = \sqrt{7}.$$

Find the value of $y(3)$.

Answer 1(a)	5

(Show your working below and on the next page.)

$$y\,dy = x^2\,dx$$

$$\frac{1}{2}y^2 = \frac{1}{3}x^3 + C$$

$$\frac{1}{2}(\sqrt{7})^2 = \frac{1}{3}(0)^3 + C$$

$$C = \frac{7}{2}$$

$$y^2 = \frac{2}{3}x^3 + 7$$

$$[y(3)]^2 = \frac{2}{3}(3^3) + 7$$

$$= 25$$

$$y(3) = \underline{\underline{5}}$$

Question 1 (b) [5 marks]

Glucose is added intravenously to the bloodstream at a rate of 5 units per minute, and the body removes glucose from the bloodstream at a rate proportional to the amount of glucose present. Initially there are 60 units of glucose in the bloodstream. After 2 minutes, there are w units of glucose in the bloodstream. If after a very long time, there are 10 units of glucose in the bloodstream, what is the value of w ? Give your answer correct to two decimal places.

Answer 1(b)	28.39

(Show your working below and on the next page.)

$$\frac{dx}{dt} = 5 - kx = -k\left(x - \frac{5}{k}\right), \quad k>0.$$

$$\Rightarrow \ln\left|x - \frac{5}{k}\right| = -kt + C \Rightarrow x - \frac{5}{k} = Ae^{-kt}$$

$$x(0) = 60 \Rightarrow A = 60 - \frac{5}{k}$$

$$\therefore x = \frac{5}{k} + \left(60 - \frac{5}{k}\right)e^{-kt}$$

$$\lim_{t\to\infty} x = 10 \Rightarrow \frac{5}{k} = 10 \Rightarrow k = \frac{1}{2}$$

$$\therefore x = 10 + 50e^{-\frac{1}{2}t}$$

$$x(2) = 10 + 50e^{-1} \approx \underline{\underline{28.39}}$$

Question 2 (a) [5 marks]

Let $y(x)$ be a solution of the initial value problem

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x^2}y^3, \quad x > 0, \quad \text{and } [\,y(1)]^2 = \frac{5}{7}.$$

Find the **exact value** of $[y(2)]^2$.

Answer 2(a)	$\frac{5}{81}$

(Show your working below and on the next page.)

Let $z = y^{1-3} = y^{-2} \Rightarrow dz = -2y^{-3}dy$

$$\frac{-\frac{1}{2}y^3 dz}{dx} + \frac{2}{x}y = \frac{1}{x^2}y^3$$

$$\frac{dz}{dx} - \frac{4}{x}z = -\frac{2}{x^2}$$

$$e^{\int -\frac{4}{x}dx} = e^{-4\ln x} = \frac{1}{x^4}$$

$$\therefore z = x^4 \int \frac{1}{x^4}\left(-\frac{2}{x^2}\right) dx = x^4 \left\{ \frac{2}{5}\frac{1}{x^5} + c \right\}$$

$$\therefore \frac{1}{y^2} = \frac{2}{5x} + cx^4$$

$$[y(1)]^2 = \frac{5}{7} \Rightarrow \frac{7}{5} = \frac{2}{5} + c \Rightarrow c = 1$$

$$\therefore y^2 = \frac{1}{\frac{2}{5x} + x^4}$$

$$[y(2)]^2 = \frac{1}{\frac{1}{5} + 16} = \underline{\underline{\frac{5}{81}}}$$

Question 2 (b) [5 marks]

The growth of rabbits in your rabbit farm followed a logistic population model with a birth rate per capita of 10 rabbits per rabbit per year. You observed that their number had approached to a logistic equilibrium population of 2500 rabbits. One day your friend Dr. Good visited your farm and suggested that you try to mix some of his latest invention of Vitamin X into your rabbit feed to boost the reproduction rate. You followed his suggestion and after a long period of time, observed that the rabbit population had reached a new logistic equilibrium of 3000 rabbits. If the new rabbit birth rate per capita after Vitamin X was introduced was B rabbits per rabbit per year, what is the value of B ?

Answer 2(b)	12

(Show your working below and on the next page.)

$$\frac{10}{s} = 2500 \Rightarrow s = \frac{10}{2500} = \frac{1}{250}$$

$$\frac{B}{s} = 3000 \Rightarrow B = 3000/250$$

$$= \underline{\underline{12}}$$

Question 3 (a) [5 marks]

An oil drum in the form of a 4 feet long right circular cylinder is in equilibrium when it floats upright with its axis vertical and half submerged in a lake. The oil drum is then further submerged so that it extends 1 foot above water, and then released from rest at time $t = 0$. Assume that there is no friction between the sides of the drum and the water in the lake as the drum moves up and down in the lake and that the gravitational constant equals to 32 ft/sec^2, find the **exact value**, in feet, of the length of the part of the drum that is **above** water at time $t = \frac{\pi}{12}$ second.

Answer 3(a)	1.5

(Show your working below and on the next page.)

Let x = length above water (in feet) at time t.

ρ = density of water, A = cross sectional area

$\therefore mg = \rho 2Ag \Rightarrow m = 2\rho A$

$m\ddot{x} = \rho(4-x)Ag - mg = 4\rho Ag - x\rho Ag - 2\rho Ag$

$2\rho A\ddot{x} = 2\rho Ag - x\rho Ag \Rightarrow \ddot{x} = g - \frac{x}{2}g$

$\therefore \ddot{x} + 16x = 32$

$\therefore x = C_1 \cos 4t + C_2 \sin 4t + 2$

$x(0) = 1,\ \dot{x}(0) = 0 \Rightarrow C_1 = -1,\ C_2 = 0$

$\therefore x = 2 - \cos 4t$

$x(\frac{\pi}{12}) = 2 - \cos\frac{\pi}{3} = 2 - \frac{1}{2} = \underline{\underline{1.5}}$

Question 3 (b) [5 marks]

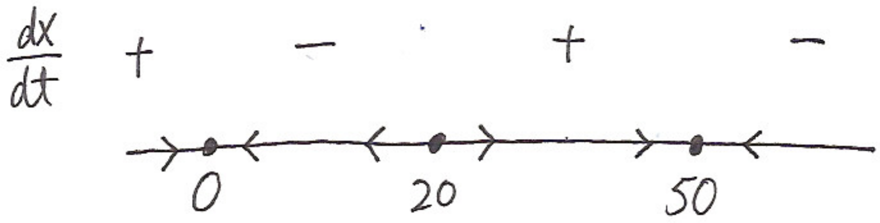
The population x of a rare species of monkey is governed by the equation

$$\frac{dx}{dt} = -\frac{11}{8}x\,(20 - x)\left(1 - \frac{1}{50}x\right).$$

Initially there are q monkeys. (i) If $q = 19$, what will the monkey population eventually be, after a very long time? (ii) If $q = 157$, what will the monkey population eventually be, after a very long time?

Answer 3(b)(i)	0	**Answer 3(b)(ii)**	50

(Show your working below and on the next page.)



Question 4 (a) [5 marks]

A cantilevered beam of length L has a weight per unit length given by

$$\frac{2ax}{L},$$

where a is constant and x measures distance from the point of attachment. It is horizontal at the end where it is attached to a wall. Find the maximum deflection at $x = L$. (Hint: you may use the formula $\frac{d^4y}{dx^4} = \frac{w(x)}{EI}$, where $w(x)$ is the force per unit length acting on the beam and is positive in the upward direction, and the fact that $\frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = 0$ at $x = L$.) Give your answer in terms of a, L, E and I.

Answer 4(a)	$-\frac{11}{60}\left(\frac{aL^4}{EI}\right)$

(Show your working below and on the next page.)

$$\frac{d^4y}{dx^4} = -\frac{2ax}{EIL} \Rightarrow \frac{d^3y}{dx^3} = -\frac{ax^2}{EIL} + C_1$$

$$0 = -\frac{aL^2}{EIL} + C_1 \Rightarrow \frac{d^3y}{dx^3} = -\frac{ax^2}{EIL} + \frac{aL}{EI}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{ax^3}{3EIL} + \frac{aL}{EI}x + C_2$$

$$0 = -\frac{aL^3}{3EIL} + \frac{aL^2}{EI} + C_2 \Rightarrow \frac{d^2y}{dx^2} = -\frac{ax^3}{3EIL} + \frac{aLx}{EI} - \frac{2aL^2}{3EI}$$

$$\frac{dy}{dx}(0) = 0 \Rightarrow \frac{dy}{dx} = -\frac{ax^4}{12EIL} + \frac{aLx^2}{2EI} - \frac{2aL^2x}{3EI}$$

$$y(0) = 0 \Rightarrow y = -\frac{ax^5}{60EIL} + \frac{aLx^3}{6EI} - \frac{aL^2x^2}{3EI}$$

$$\therefore \Delta = -\frac{aL^4}{60EI} + \frac{aL^4}{6EI} - \frac{aL^4}{3EI} = \underline{\underline{-\frac{11}{60}\left(\frac{aL^4}{EI}\right)}}$$

Question 4 (b) [5 marks]

Use Laplace transform to solve the following initial value problem

$$y'' - y' = te^t + 1, \; y(0) = 1, \; y'(0) = 0.$$

(Note: Zero mark if you do not use Laplace transform.)

Answer 4(b)	$y = -1 - t + 2e^t - te^t + \frac{1}{2}t^2e^t$

(Show your working below and on the next page.)

Let $L(y) = Y$

$\therefore s^2Y - s - sY + 1 = \frac{1}{(s-1)^2} + \frac{1}{s}$

$\therefore s(s-1)Y = s - 1 + \frac{1}{(s-1)^2} + \frac{1}{s}$

$\therefore Y = \frac{1}{s} + \frac{1}{s(s-1)^3} + \frac{1}{s^2(s-1)}$

$= \frac{1}{s} + \left\{ -\frac{1}{s} + \frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{(s-1)^3} \right\}$

$+ \left\{ -\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s-1} \right\}$

$= -\frac{1}{s} - \frac{1}{s^2} + \frac{2}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{(s-1)^3}$

$\therefore \underline{\underline{y = -1 - t + 2e^t - te^t + \frac{1}{2}t^2e^t}}$

Question 5 (a) [5 marks]

In an RLC circuit of inductance L henrys, resistance R ohms, capacitance C farads and voltage V volts, it is known that the electric current I satisfies the equation

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(u)\,du = V.$$

At time $t = 0$ seconds, seeing that there is no voltage applied to the circuit and that $I = 0$ at that time, you immediately turn the switch on and off , firing a short burst of voltage into it. You observe that the resulting current at $t > 0$ satisfies

$$I(t) = e^{-3t}\cos t - 3e^{-3t}\sin t.$$

If $L = \frac{1}{2}$, find the value of R and C.
(Hint: You may use the formula $\pounds\left(\int_0^t I(u)\,du\right) = \frac{1}{s}\pounds(I(t))$, where $\pounds$ denotes the Laplace transform.)

Answer 5(a) R=	3	Answer 5(a) C=	$\frac{1}{5}$

(Show your working below and on the next page.)

$$LS\mathcal{L}(I) + R\mathcal{L}(I) + \frac{1}{SC}\mathcal{L}(I) = \mathcal{L}(A\delta(t)) = A$$

$$\therefore \mathcal{L}(I) = \frac{A}{LS+R+\frac{1}{SC}} = \frac{AS}{LS^2+RS+1/C} = \frac{AS}{\frac{1}{2}S^2+RS+1/C}$$

$$\text{next } \mathcal{L}(I) = \mathcal{L}\{e^{-3t}\cos t - 3e^{-3t}\sin t\} = \frac{S+3}{(S+3)^2+1} - 3\frac{1}{(S+3)^2+1}$$

$$= \frac{S}{S^2+6S+10} = \frac{AS}{AS^2+6AS+10A}$$

$$\therefore A = \frac{1}{2},\ R = 6A = \underline{\underline{3}},\ \frac{1}{C} = 10A = 5 \Rightarrow C = \underline{\underline{\frac{1}{5}}}$$

Question 5 (b) [5 marks]

The growth of a certain fish population $N(t)$ is governed by

$$\frac{dN}{dt} = 0.01N,$$

where time is measued in days. Initially at Day Zero, $N(0) = 1000$ tons. Harvesting is allowed for a 30 days period beginning with Day Zero at the rate of 20 tons per day. This can be modeled by the equation

$$\frac{dN}{dt} = 0.01N - 20\left[1 - u(t-30)\right],$$

where u is the unit step function. Find the amount of fish in tons (i) at time $t = 25$; and (ii) at time $t = 50$. Give your answers correct to two decimal places.

Answer 5(b) (i)	715.97	Answer 5(b) (ii)	794.08

(Show your working below and on the next page.)

$$sL(N) - 1000 = 0.01\,L(N) - 20\left\{\frac{1}{s} - \frac{e^{-30s}}{s}\right\}$$

$$\therefore L(N) = \frac{1000}{s-0.01} - \frac{20}{s(s-0.01)} + \frac{20}{s(s-0.01)}e^{-30s}$$

$$= \frac{1000}{s-0.01} - 2000\left\{\frac{1}{s-0.01} - \frac{1}{s}\right\} + 2000\left\{\frac{1}{s-0.01} - \frac{1}{s}\right\}e^{-30s}$$

$$\therefore N = 1000e^{0.01t} - 2000e^{0.01t} + 2000 + 2000\left\{e^{0.01(t-30)} - 1\right\}u(t-30)$$

$$= 2000 - 1000e^{0.01t} + 2000\left\{e^{0.01(t-30)} - 1\right\}u(t-30)$$

(i) $N(25) = 2000 - 1000e^{0.25} \approx 715.97$

(ii) $N(50) = 2000 - 1000e^{0.5} + 2000\left\{e^{0.2} - 1\right\} \approx \underline{794.08}$

Question 6 (a) [5 marks]

The probability that a smoker will quit smoking and become a nonsmoker a year later is 30%. The probability a nonsmoker will become a smoker a year later is 10%. We use the following transition matrix to represent this information:

$$\begin{pmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{pmatrix}.$$

Mr. Tan is currently a smoker. What is the probability that he becomes a nonsmoker three years from now ? Give your answer correct to three decimal places.

Answer 6(a)	0.588

(Show your working below and on the next page.)

$$\begin{pmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{pmatrix}^2 = \begin{pmatrix} 0.52 & 0.16 \\ 0.48 & 0.84 \end{pmatrix}$$

$$\begin{pmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{pmatrix}^3 = \begin{pmatrix} 0.52 & 0.16 \\ 0.48 & 0.84 \end{pmatrix}\begin{pmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{pmatrix}$$

$$= \begin{pmatrix} * & * \\ 0.588 & * \end{pmatrix}$$

Question 6 (b) [5 marks]

Let A denote the matrix

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}.$$

Find the **exact expression** of the matrix e^A.

Answer 6(b)	$\begin{pmatrix} \frac{1}{2}e^{-1}+\frac{1}{2}e^{3} & -e^{-1}+e^{3} \\ -\frac{1}{4}e^{-1}+\frac{1}{4}e^{3} & \frac{1}{2}e^{-1}+\frac{1}{2}e^{3} \end{pmatrix}$

(Show your working below and on the next page.)

$$\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda - 3 = 0 \Rightarrow \lambda = -1, 3.$$

$\lambda = -1 \Rightarrow x + 2y = 0 \Rightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is an eigenvector

$\lambda = 3 \Rightarrow x - 2y = 0 \Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector

$$P = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \Rightarrow P^{-1} = \frac{1}{4}\begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}$$

$$\therefore A = P \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} P^{-1}$$

$$e^A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-1} & 0 \\ 0 & e^3 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}e^{-1}+\frac{1}{2}e^{3} & -e^{-1}+e^{3} \\ -\frac{1}{4}e^{-1}+\frac{1}{4}e^{3} & \frac{1}{2}e^{-1}+\frac{1}{2}e^{3} \end{pmatrix}$$

Question 7 (a) [5 marks]

Find the new coordinates of the point $(2, 1)$ if we shear 45^o parallel to the x axis, and then rotate 135^o anticlockwise. You may either give exact values for your answer or give your answer correct to one decimal place.

Answer 7(a)	$(-2\sqrt{2}, \sqrt{2})$ or $(-2.8, 1.4)$

(Show your working below and on the next page.)

Shear 45°: $S = \begin{pmatrix} 1 & \tan 45^\circ \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Rotate 135°: $R = \begin{pmatrix} \cos 135^\circ & -\sin 135^\circ \\ \sin 135^\circ & \cos 135^\circ \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2\sqrt{2} \\ \sqrt{2} \end{pmatrix}$$

Ans. $\underline{\underline{(-2\sqrt{2}, \sqrt{2}) \simeq (-2.8, 1.4)}}$

Question 7 (b) [5 marks]

Classify the following system of differential equations (that is, say whether it represents a nodal source, spiral source, etc etc.)

$$\begin{cases} \frac{dx}{dt} = -2x - 5y \\ \frac{dy}{dt} = x - 3y \end{cases}$$

and then carefully and clearly draw its phase plane diagram in the space below.

Answer 7(b)	SPIRAL SINK

(Show your working and drawing below and on the next page.)

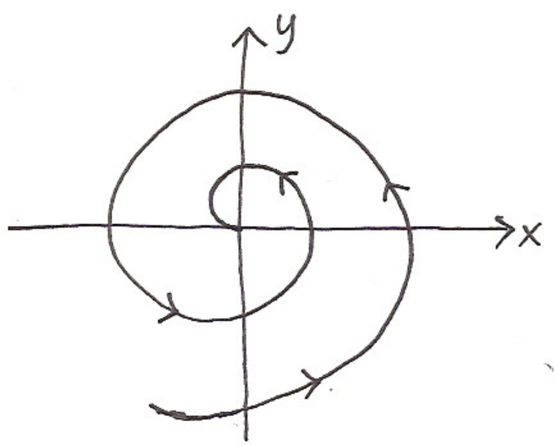
$\det = 6 + 5 = 11$

$\mathrm{Tr} = -2 + (-3) = -5$

$(\mathrm{Tr})^2 - 4\det = 25 - 44 = -\mathrm{ve}$

$\therefore$ spiral sink

at $(\alpha, 0)$ where $\alpha > 0$, $\frac{dy}{dt} = \alpha = +\mathrm{ve} \Rightarrow$ graph goes $\uparrow$



Question 8 (a) [5 marks]

Solve the following system of differential equations:

$$\begin{cases} \frac{dx}{dt} = -4x + 3y \\ \frac{dy}{dt} = -2x + y \end{cases}$$

with $x(0) = 4$ and $y(0) = 3$.

Answer 8(a)	$\begin{cases} x = 3e^{-2t} + e^{-t} \\ y = 2e^{-2t} + e^{-t} \end{cases}$

(Show your working below and on the next page.)

$$\begin{vmatrix} -4-\lambda & 3 \\ -2 & 1-\lambda \end{vmatrix} = (-4-\lambda)(1-\lambda) + 6 = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$$

$\therefore \lambda = -2, -1.$

$\lambda = -2 \Rightarrow -2x + 3y = 0 \Rightarrow \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is an eigenvector

$\lambda = -1 \Rightarrow -2x + 2y = 0 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-2t} \begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3c_1 e^{-2t} + c_2 e^{-t} \\ 2c_1 e^{-2t} + c_2 e^{-t} \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 3c_1 + c_2 \\ 2c_1 + c_2 \end{pmatrix} \Rightarrow c_1 = 1, \ c_2 = 1$$

$$\therefore \begin{cases} x = 3e^{-2t} + e^{-t} \\ y = 2e^{-2t} + e^{-t} \end{cases}$$

Question 8 (b) [5 marks]

The orcs of Saruman have gone to war against the orcs of Sauron. Both sides use diseases against each other, as well as conventional weapons. Suppose that we model this situation using the system of ordinary differential equations

$$\begin{cases} \frac{dx}{dt} = -4x - 3y \\ \frac{dy}{dt} = -x - 2y \end{cases}$$

where x and y denote the number of Saruman's orcs and Sauron's orcs respectively at any time t. At time $t = 0$, Saruman sends 15000 orcs against an army of k Sauron orcs. What is the minimum positive integer N that k **must exceed** (i.e. $k > N$) in order that all Saruman orcs are killed in the battle and some Sauron orcs will survive the battle ?

Answer 8(b)	5000

(Show your working below and on the next page.)

$\begin{vmatrix} -4-\lambda & -3 \\ -1 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (4+\lambda)(2+\lambda) - 3 = 0 \Rightarrow \lambda^2 + 6\lambda + 5 = 0 \Rightarrow \lambda = -1, -5.$

$\lambda = -1 \Rightarrow -x - y = 0 \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector

$\lambda = -5 \Rightarrow x - 3y = 0 \Rightarrow \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector

$(15000, k)$ lies above $x - 3y = 0$

$\Rightarrow 3k > 15000$

$\Rightarrow \underline{\underline{k > 5000}}$