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AUGUST 2009 ANALYSIS

1.) Yes, we can conclude it.
We prove by contradiction, suppose $f(x_0) \neq g(x_0)$, then we can assume that $f(x_0) > g(x_0)$.
Then by the monotone decreasing property of f we have
$\{x \in (0,1): f(x) \geq f(x_0)\} \supset (0, x_0)$, so $|\{x \in (0,1): f(x) \geq f(x_0)\}| > x_0$
On the other hand, by the left continuity of g, we must have an $\delta > 0$ s.t. for $x \in (x_0 - \delta, x_0)$ we have $f(x) \leq f(x_0)$. Therefore, using the monotone decreasing g,
$\{x \in (0,1): g(x) \geq f(x_0)\} \subset (0, x_0 - \delta)$, so $|\{x \in (0,1): g(x) \geq g(x_0)\}| < x_0 - \delta$.
But this contradicts the original assumption, so $f(x) = g(x)$ for $\forall x \in (0,1)$.

2.) $S = \{(x,y,z) \in \mathbb{R}^3 : \begin{pmatrix} x \\ y \\ z \end{pmatrix}^T \underline{A}^T \underline{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \leq 1\}$, where $\underline{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

To calculate $|S|$, we can use the Theorem of linear transformations, and use the $\underline{A}$ transformation on it, then $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underline{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and

$S' = \{(x', y', z') \in \mathbb{R}^3 : \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}^T \cdot \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \leq 1\}$. This is a sphere of radius one, so $|S'| = \frac{4}{3}\pi$. On the other hand, by the Theorem of linear transformations, $|S'| = \det A \cdot |S|$, so $|S| = \frac{4}{3}\pi \cdot (\det \underline{A})^{-1}$.

3.) First let's note $D_0 = \{x \in D : \phi(x,t)$ is cont. in $t\}$, then $|D - D_0| = 0$, so we can limit our discussion to D_0.
Let's define the function $\delta_\varepsilon(x) = \sup\{\delta \in \mathbb{R}^+, \phi(x, f(x) + [-\delta, \delta]) \subset [\phi(x,f(x)) - \varepsilon, \phi(x,f(x)) + \varepsilon]\}$
i.e. the "factor of continuousity of $\phi(x,t)$ in t at point x.
Then we know that by continuousity of $\phi(x,t)$ in t for $\forall x \in D_0$, we have $\delta_\varepsilon(x) > 0$ for $\forall x \in D_0$. Then we can write
$\{x \in D_0 : |\phi(x, f_n(x)) - \phi(x, f(x))| < \varepsilon\} \subset \{x \in D_0 : |f(x) - f_n(x)| > \delta\} \cup \{x \in D_0 : \delta_\varepsilon(x) < \delta\}$

As $n \to \infty$, by the convergence in measure of f, $|\{x \in D_0 : |f(x) - f_n(x)| > \delta\}| \to 0$ for $\forall \delta > 0$, and as we take $\delta \to 0$, $|\{x \in D_0 : \delta_\varepsilon(x) < \delta\}| \to 0$, so $\{\phi(x, f_n(x))\} \to \phi(x, f(x))$ in measure on D_0, and then trivially on D too.

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9.) On $L^2[0,\pi]$, we can define the scalar product of functions g,h as

$\langle g,h\rangle = \int_0^\pi g\cdot h\,dx$, and the norm of h as

$\|h\| = \left(\int_0^\pi h^2\,dx\right)^{1/2}$.

Then if we note $\sin x \equiv g(x)$ and $\cos x \equiv h(x)$, then $g,h\in L^2[0,\pi]$ obviously, and the claims of the question mean $\|f-g\|\le \frac{2}{3}$, $\|f-h\|\le\frac{1}{3}$. But then the triangle inequality implies that $\|g-h\|\le \frac{2}{3}+\frac{1}{3}=1$.

In fact,

$$\|g-h\| = \left(\int_0^\pi (\sin x-\cos x)^2\,dx\right)^{1/2} = \left[\int_0^\pi \sin^2 x+\cos^2 x-2\sin x\cos x\,dx\right]^{1/2}=\sqrt{\pi}$$

But $\sqrt{\pi}>1$, so there can't be such an $f\in L^2[0,\pi]$.

10.) Let's take the logarithm of both sides (they're non-zeroes):

$$\frac{1}{n}\sum_{k=1}^{n} \ln\left(\frac{\sin a_k}{a_k}\right) \le \ln\left(\frac{\sin\alpha}{\alpha}\right)$$

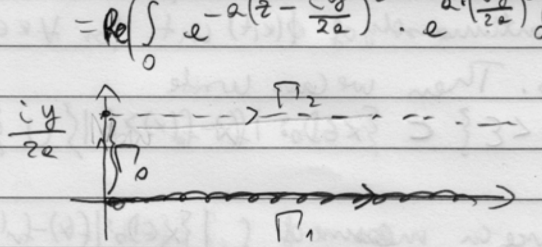
This is almost the Jensen-inequality for the function $f(x)=\ln\left(\frac{\sin x}{x}\right)$, to use it, we only need to show that it is concave. Indeed on $\alpha_k\in(0,\pi)$,

$$f'(x) = \left(\frac{x}{\sin x}\right)\cdot\left(\frac{(\cos x)x-\sin x}{x^2}\right) = \frac{\cos x}{\sin x}-\frac{1}{x}$$

$$f''(x) = \frac{-\sin^2 x-\cos^2 x}{\sin^2 x}+\frac{1}{x^2} = \frac{1}{x^2}-\frac{1}{\sin^2 x}<0$$, therefore $f(x)$ is concave, so the inequality holds.

4.) $I(y)=\int_0^\infty e^{-ax^2}\cos(yx)\,dx = \int_0^\infty \mathrm{Re}\left(e^{-az^2}e^{iyz}\,dz\right) =$

$$=\mathrm{Re}\left(\int_0^\infty e^{-a\left(z-\frac{iy}{2a}\right)^2}\cdot e^{a\cdot\left(\frac{iy}{2a}\right)^2}\,dz\right) = \mathrm{Re}\left(\underbrace{\int_0^\infty e^{-a\left(z-\frac{iy}{2a}\right)^2}\,dz}_{f(z)}\right)\cdot e^{-\frac{y^2}{4a}}.$$



$f(z)$ is entire, and $|f(z)|\to 0$ as $\mathrm{Re}(z)\to+\infty$, so

$-\int_{\Gamma_1} f(z)\,dz+\int_{\Gamma_0}f(z)\,dz+\int_{\Gamma_2}f(z)\,dz=0$, and in particular

$$\mathrm{Re}\int_{\Gamma_1}f(z)\,dz = \mathrm{Re}\int_{\Gamma_0}f(z)\,dz + \mathrm{Re}\int_{\Gamma_2}f(z)\,dz$$

$\frac{1}{\sqrt{\pi}} \cdot e^{-x}$

But $f(z)$ is real on $z \in \Gamma_0$ and dz is imaginary on $z \in \Gamma_0$,
so $\mathrm{Re} \int_{\Gamma_0} f(z)\,dz = 0$, and therefore

$$\mathrm{Re} \int_{\Gamma_1} f(z)\,dz = \mathrm{Re} \int_{\Gamma_2} f(z)\,dz = \int_0^\infty e^{-az^2}\,dz = \int_0^\infty e^{-(z\cdot\sqrt{a})^2} \frac{dz\cdot\sqrt{a}}{\sqrt{a}} =$$

$$= \frac{1}{\sqrt{a}} \cdot \sqrt{\frac{\pi}{2}} = \sqrt{\frac{\pi}{2a}}.$$

So $I(y) = \sqrt{\frac{\pi}{2a}} \cdot e^{-\frac{y^2}{4a}}$

5.) Firstly, we see that for z real,

$$\mathrm{Im}(f(z)) = \mathrm{Im}\left(\frac{z}{a - e^{-iz}}\right) = \mathrm{Im}\left(\frac{z(a - e^{iz})}{(a - e^{-iz})(a - e^{iz})}\right) =$$

$$= \frac{-z \sin z}{a^2 + 1 - 2a\cos z}$$

Therefore $I = \int_0^\pi \frac{x \sin x}{1 + a^2 - 2a\cos x}\,dx = -\mathrm{Im}\left(\int_0^\pi f(z)\,dz\right).$

Now in order to calculate this integral, we need to find the singularities of $f(z) = \frac{z}{a - e^{-iz}}$, i.e. the points where $a = e^{-iz}$.
These have the form $z_k = i \ln a + k 2\pi$ with $k \in \mathbb{Z}$.
For symmetrical reasons, we can write

$$I = -\frac{1}{2} \cdot \mathrm{Im}\left(\int_{-\pi}^{\pi} f(z)\,dz\right).$$