NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2008-2009

MA5213 Advanced Partial Differential Equations

May 2009 — Time allowed : TWO and HALF hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages.

2. Select **FOUR** questions to answer.

3. This is a closed book examination.

1. (25 marks)

   (i) (10 marks) Find the **explicit formula** of the Green's function $G(x,t;y)$ for the initial-boundary value problem

$$\begin{cases} (\partial_t + \partial_x - \partial_x^2)\mathbb{G}(x,t;y) = 0 \text{ for } x,t,y > 0, \\ \mathbb{G}(0,t;y) = 0 \text{ for } t > 0, y > 0, \\ \mathbb{G}(x,0;y) = \delta(x-y). \end{cases}$$

   (ii) (15 marks) Use the Green's function to show there exists $C > 0$ such that the solution $u(x,t)$ of the nonlinear problem

$$\begin{cases} (\partial_t + \partial_x - \partial_x^2)u(x,t) = -\frac{1}{2}\partial_x[u(x,t)^2] \text{ for } x,t > 0, \\ u(0,t) = 0, \\ |u(x,0)| \le \epsilon e^{-2x} \text{ for } x > 0, \epsilon \ll 1 \end{cases}$$

   satisfies for all $x,t > 0$

$$|u(x,t)| \le O(1)\epsilon \left(\frac{e^{-\frac{(x-t)^2}{6t}}}{\sqrt{1+t}} + e^{-(x+t)/C} \right).$$

2. (25 marks)
   Let $\mathbb{G}(x,t)$ be the Green's function:

$$\begin{cases} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \right) \mathbb{G} = 0 \text{ for } x \in \mathbb{R}, t > 0; \\ \mathbb{G}(x,0) = \delta(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{cases}$$

   Show that there exists $C > 0$ such that

$$\left| \mathbb{G}(x,t) - \begin{pmatrix} \frac{e^{-3t/2}\delta(x-t) + e^{-t/2}\delta(x+t)}{2} & \frac{e^{-3t/2}\delta(x-t) - e^{-t/2}\delta(x+t)}{2} \\ \frac{e^{-3t/2}\delta(x-t) - e^{-t/2}\delta(x+t)}{2} & \frac{e^{-3t/2}\delta(x-t) + e^{-t/2}\delta(x+t)}{2} \end{pmatrix} \right|$$
$$\le O(1) \left(\frac{e^{-\frac{(x+\frac{1}{2}t)^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-\frac{|x|+t}{C}} \right).$$

3. (25 marks)
Let $\mathbb{G}_b(x,t;y)$ be the Green's function of an initial-boundary value problem:

$$\begin{cases} \left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\partial_t + \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\partial_x + \begin{pmatrix}0 & 0\\ 1 & 2\end{pmatrix}\right)\mathbb{G}_b(x,t;y) = 0 \text{ for } x,y,t>0;\\ \mathbb{G}_b(x,0;y) = \delta(x-y)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\\ (1,1)\mathbb{G}_b(0,t;y) = (0,0) \text{ for all } y,t\geq 0. \end{cases}$$

(i) (10 marks) Show that the function $\mathbb{G}_b(x,t-s;y)$ satisfies the backward equation

$$\begin{cases} -\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\partial_s\mathbb{G}_b - \partial_y\mathbb{G}_b\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \mathbb{G}_b\begin{pmatrix}0 & 0\\ 1 & 2\end{pmatrix} = 0 \text{ for } x,y,t>0;\\ \mathbb{G}_b(x,0;y) = \delta(x-y)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\\ \mathbb{G}_b(x,t;0)\begin{pmatrix}1\\ -1\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \text{ for all } y,t\geq 0. \end{cases}$$

(ii) (15 marks) Show that there exists $C>0$ such that

$$|\mathbb{G}(x,t;y) - \mathbb{G}_b(x,t;y)| \leq Ce^{-(x+t)/C} \text{ for } x,t>0, y\in[0,1],$$

where $\mathbb{G}(x,t;y)$ is the Green's function constructed in Problem 2.

4. (25 marks) Consider a hyperbolic model for the porous media equation

$$\begin{cases} \rho_t + m_x = 0,\\ m_t + (m^2/\rho + \rho^2)_x = -m. \end{cases}$$

The state $(\rho_*, m_*) = (1,0)$ is a trivial solution.
Assume that the initial data $\rho(\cdot,0)$ and $m(\cdot,0)$ satisfying

$$\|\rho(\cdot,0)-1\|_{H^3}, \|m(\cdot,0)\|_{H^3} \leq \epsilon \ll 1.$$

Show by energy estimates that there exists $C>0$ such that

$$\|\rho(\cdot,t)-1\|_{H^3}, \|m(\cdot,t)\|_{H^3} \leq C\epsilon \text{ for all } t>0.$$

5. (25 marks)
Consider the viscous Burgers' equation $u_t + uu_x - u_{xx} = 0$. The function $\phi(x) = -\tanh\frac{x}{2}$ is a stationary viscous shock profile connecting a shock wave $(1, -1)$.
Assume the initial data of $u(x,t)$ satisfying

$$|u(x,0) - \phi(x)| \leq \epsilon e^{-|x|/8}, \int_{\mathbb{R}} (u(x,0) - \phi(x))dx = 0$$

and

$$0 < \epsilon \ll 1.$$

Show that there exists $C > 0$ such that

$$|u(x,t) - \phi(x)| \leq C\epsilon e^{-|x|/4} e^{-3t/16} \text{ for } t > 0.$$

END OF PAPER