NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 2008-2009

PhD QUALIFYING EXAMINATION

PAPER 1

ALGEBRA

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions.
3. This is an open-book examination. Any theorems used without proof should be stated clearly.

Question 1. If you use Zorn's Lemma, then its use should be made clear.

In a group G, let $[g, h] = ghg^{-1}h^{-1}$ denote the *commutator* of elements g and h, and let the *commutator subgroup* $[G, G]$ be the subgroup of G generated by the commutators of G. A group P is called *perfect* if $P = [P, P]$.

(a) Prove that if a group contains only countably many perfect subgroups, then it contains subgroups that are maximal among all countable perfect subgroups.

(b) Prove that every group G contains a unique maximum perfect subgroup $\mathcal{P}G$, and that $\mathcal{P}G$ is

(i) normal in G; and

(ii) the intersection of the transfinite derived series of G, where the iterated commutator subgroup $G^{(\alpha)}$ is defined as:

- $[G^{(\beta)}, G^{(\beta)}]$ if $\alpha = \beta + 1$ is a successor ordinal; and
- $\bigcap_{\gamma<\alpha} G^{(\gamma)}$ if α is a limit ordinal.

Question 2. Construct the character tables for all 5 (isomorphism types of) groups of order 8, and show that there are only 4 distinct character tables among these 5 groups.

Question 3. Consider the category CommRing of commutative rings (with 1) and ring homomorphisms (preserving 1).

(a) Show that this category admits coproducts.

(b) Denoting the coproduct of R and S by $R \odot S$, give four **distinct** nonzero commutative rings A, B, C, D such that:

(i) for all R, $R \odot A \cong R$;

(ii) $B \odot C \cong 0$, where 0 denotes the zero ring with a unique element;

(iii) $D \odot D \cong D$.

Question 4. Let $p(X) = (X^3 - 2)(X^3 - 3)$.

(a) Find the lattices of

(i) all normal extension fields of $\mathbb{Q}$ contained in the splitting field E of $p(X)$ over $\mathbb{Q}$;

(ii) all normal subgroups of the Galois group of E over $\mathbb{Q}$.

(b) Over $\mathbb{F}_5$, what is the Galois group of the splitting field of $p(X)$?

Question 5. Let $\mathrm{Symm}_n(R)$ be the set of symmetric $n \times n$ matrices over a commutative ring R (with 1).

(a) Show that, with the natural scalar multiplication, $\mathrm{Symm}_n(R)$ is a free left R-module. What is its rank?

(b) Now assume that 2 is invertible in R. Provide a nontrivial action that makes $\mathrm{Symm}_n(R)$ a projective left module over the ring $M_n(R)$ of all $n \times n$ matrices over R, and prove that it is projective. By taking R to be finite, or otherwise, show that this module need not be free.

END OF PAPER