NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 2008-2009

Ph.D. QUALIFYING EXAMINATION

PAPER 1

ALGEBRA

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions.

Ph.D. Qualifying Examination
Year 2008/2009, Semester 1

ALGEBRA

Answer all questions. Each question carries 20 marks.

(1) (a) Let G be a finite simple group, and suppose that H is a proper subgroup of G of index k. Show that there exists an injective group homomorphism from G to the alternating group A_k of degree k.
(b) Show that a group of order 120 is not simple.

(2) Let V be a finite-dimensional vector space of an algebraically closed field F of positive characteristic p. Let $\alpha : V \to V$ be a linear operator on V, and suppose that there exists a positive integer n such that $\alpha^n(v) = v$ for all $v \in V$, while for each positive integer i less than n, there exists $v_i \in V$ such that $\alpha^i(v_i) \neq v_i$. Show that α is diagonalisable if and only if n is not divisible by p.

(3) (a) Let R and S be integral domains with $R \subseteq S$. Prove or disprove the following:
(i) If R is a Euclidean domain, then S is a unique factorisation domain.
(ii) If S is a Euclidean domain, then R is a unique factorisation domain.
(b) Let $\phi : T \to U$ be a surjective ring homomorphism between two integral domains T and U. Prove or disprove the following:
(i) If T is a principal ideal domain, then U is a principal ideal domain.
(ii) If T is a unique factorisation domain, then U is a unique factorisation domain.

(4) Let K be the splitting field of $X^4 - 2$ over the field $\mathbb{Q}$ of rational numbers.
(a) Show that there exist field automorphisms τ and σ of K satisfying the following properties:
- τ has order 2;
- σ has order 4;
- $\tau \circ \sigma = \sigma^{-1} \circ \tau$.

(b) Hence, or otherwise, find all intermediate fields between $\mathbb{Q}$ and K.

(5) Let R be a ring with multiplicative identity, and let M be a finitely generated left R-module.
(a) Let B be a non-empty finite subset of M. Show that M is a free R-module with basis B if and only if every function from B to any left R-module N can be uniquely extended to a left R-module homomorphism from M to N.
(b) Suppose further that R is a principal ideal domain. Prove that M is a free R-module if and only if M is a projective R-module.

END OF PAPER