NATIONAL UNIVERSITY OF SINGAPORE

Faculty of Science

Semester 2 Examination 2007/08

MA5213 Advanced Partial Differential Equations

May 2008 — Time allowed : 2.5 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR** (4) questions and comprises **TWO** (2) printed pages.

2. Answer **ALL FOUR** questions.

3. This is a closed book examination. No books or notes may be used.

4. Candidates may use calculators. However, they should explain in detail all the steps in the calculations.

Answer all four questions.

1. (25 Marks) Consider the initial boundary value problem

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u = -(v - u^2), \end{cases}$$

for $x, t > 0$. The boundary condition is imposed by $u + v = 0$ at $x = 0$. The initial data satisfies

$$\sup_{x>0} \left(|u(x,0)| + |\partial_x u(x,0)| + |v(x,0)| + |\partial_x v(x,0)| \right) \ll 1.$$

Show that there exists $\tau_0 > 0$ such that $(u, v)(x,t)$ exists in the time interval $t \in (0, \tau_0)$.

2. (25 Marks) Consider the Burgers' equation $u_t + uu_x = 0$ with the initial data

$$u(x,0) = \begin{cases} 0 \text{ for } x < -1, \\ 1 \text{ for } x \in (-1,1), \\ 0 \text{ for } x > 1. \end{cases}$$

Analyze the solution $u(x,t)$ including finding the asymptotic of the shock wave location and give the rate of decaying to zero which is $O(1)(1+t)^{-1/2}$.
(Hint: characteristic curve method and the Rankine-Hugoniot condition.)

3. (25 Marks) Consider the viscous Burgers' equation $u_t + uu_x = u_{xx}$ with the initial data $u(x,0) = -\tanh(x/2) + v(x,0)$, where $v(x,0)$ satisfies

$$\begin{cases} \int_{-\infty}^{\infty} v(x,0)dx = 0, \\ \sum_{k=0}^{3} \int_{-\infty}^{\infty} |\partial_x^k v(x,0)|^2 dx \ll 1. \end{cases}$$

Use energy method to show that

$$\lim_{t \to \infty} |u(x,t) + \tanh(x/2)| = 0.$$

4. (25 Marks) Consider the viscous Burgers' equation $u_t + uu_x = u_{xx}$ with the initial data $u(x,0) = -\tanh(x/2) + v(x,0)$, where $v(x,0)$ satisfies

$$\begin{cases} \int_{-\infty}^{\infty} v(x,0)dx = 0, \\ |v(x,0)| \le \epsilon e^{-|x|}. \end{cases}$$

Use the Green's function to show that when $0 < \epsilon \ll 1$, the solution $u(x,t)$ satisfies

$$|u(x,t) + \tanh(x/2)| \le O(1)\epsilon e^{-\frac{|x|}{2} - \frac{t}{4}}.$$

END OF PAPER