Matriculation Number:

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2007-2008

MA1506 MATHEMATICS II

April 2008 Time allowed: 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
Marks								

Question 1 (a) [5 marks]

Let $y > 0$ be a solution of the differential equation

$$\frac{dy}{dx} = \frac{3x^2 + 1}{2y}$$

with the initial condition $y(0) = 1$. Find the value of $y(1)$.

Answer 1(a)	$\sqrt{3}$

(Show your working below and on the next page.)

$$2y\,dy = (3x^2+1)\,dx$$

$$y^2 = x^3 + x + C$$

$$y(0) = 1 \Rightarrow 1 = C$$

$$\therefore\ y^2 = x^3 + x + 1$$

$$x = 1 \Rightarrow y^2 = 3$$

$$\Rightarrow \underline{\underline{y = \sqrt{3}}} \quad (\because y > 0)$$

Question 1 (b) [5 marks]

An old object was dug up near NUS and you were asked to help to find out how old is this object. You carried out an experiment and found that the object contained 95% of the carbon-14 found in a similar present-day sample. You looked up a table from your chemistry book and found that the half-life of carbon-14 is 5730 years (i.e. it takes 5730 years for 50% of an amount of carbon-14 to decay away). Assume that carbon-14 decays at a rate proportional to the amount present, approximately how many years old is this object?

Answer 1(b)	424

(Show your working below and on the next page.)

$$\frac{dx}{dt} = -kx$$

$$\Rightarrow x = Ae^{-kt}$$

$$\tfrac{1}{2}A = Ae^{-5730k} \Rightarrow k = \frac{\ln 2}{5730} \approx 0.00012$$

$$\therefore x = Ae^{-\frac{\ln 2}{5730}t}$$

$$0.95A = Ae^{-\frac{\ln 2}{5730}t}$$

$$\Rightarrow \ln 0.95 = -\frac{\ln 2}{5730}t$$

$$\Rightarrow t = -\frac{5730 \ln 0.95}{\ln 2} \approx \underline{\underline{424}}$$

Question 2 (a) [5 marks]

Solve the differential equation

$$y'' + 4y = \cos^2 x - \sin^2 x$$

with the initial conditions

$$y(0) = 1, \quad y'(0) = 2.$$

Answer 2(a)	$y = \cos 2x + \sin 2x + \frac{1}{4} x \sin 2x$

(Show your working below and on the next page.)

$$y'' + 4y = \cos^2 x - \sin^2 x = \cos 2x$$

$$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$$

$$y = \mathrm{Re}(z) \text{ where } z'' + 4z = e^{i2x}$$

$$\text{Try } z = Axe^{i2x}$$

$$\therefore z' = Ae^{i2x} + 2iAxe^{i2x}$$

$$z'' = 4iAe^{i2x} - 4Axe^{i2x}$$

$$\therefore 4iAe^{i2x} = e^{i2x}$$

$$\therefore A = \frac{1}{4i} = -\frac{1}{4}i$$

$$\therefore z = -\frac{1}{4}ix(\cos 2x + i\sin 2x) = \frac{1}{4}x\sin 2x - \frac{1}{4}ix\cos 2x$$

$$\therefore y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{4}x \sin 2x$$

$$y(0) = 1 \Rightarrow C_1 = 1.$$

$$y'(0) = 2 \Rightarrow 2 = 2C_2 \Rightarrow C_2 = 1.$$

$$\therefore \underline{\underline{y = \cos 2x + \sin 2x + \frac{1}{4}x \sin 2x}}$$

Question 2 (b) [5 marks]

A particle moves along the x-axis in accordance with the equation of motion

$$\ddot{x} + 6\dot{x} - 16x = 0.$$

At $t = 0$ sec, the particle is at $x = 2$ m and moving to the left with a velocity of 10 m/sec. When will the particle change direction and go to the right?

Answer 2(b)	$t = 0.223$ sec.

(Show your working below and on the next page.)

$$\lambda^2 + 6\lambda - 16 = 0$$

$$(\lambda+8)(\lambda-2) = 0$$

$$\therefore \lambda = 2 \text{ or } \lambda = -8$$

$$x = C_1 e^{2t} + C_2 e^{-8t}$$

$$\dot{x} = 2C_1 e^{2t} - 8C_2 e^{-8t}$$

$$\left.\begin{array}{l} x(0) = 2 \Rightarrow C_1 + C_2 = 2 \\ \dot{x}(0) = -10 \Rightarrow 2C_1 - 8C_2 = -10 \end{array}\right\} \Rightarrow C_1 = \frac{3}{5}, \; C_2 = \frac{7}{5}$$

$$\therefore \dot{x} = \frac{6}{5} e^{2t} - \frac{56}{5} e^{-8t}$$

The particle changes direction when $\dot{x} = 0$.

$$\therefore \frac{6}{5} e^{2t} = \frac{56}{5} e^{-8t}$$

$$\Rightarrow e^{10t} = \frac{28}{3}$$

$$\Rightarrow t = \frac{1}{10} \ln \frac{28}{3} \approx \underline{\underline{0.22336}}$$

Question 3 (a) [5 marks]

A psychologist used the equation

$$\frac{dP}{dt} = \frac{1}{1+t^2}(M - 2tP)$$

to model the performance of a certain student. Here, P denotes the student's performance at any time $t \geq 0$ and M denotes a positive constant. Assume that $P = 0$ at time $t = 0$. What is the value of t when P first reaches 40% of M ?

Answer 3(b)	$\frac{1}{2}$

(Show your working below and on the next page.)

$$\frac{dP}{dt} + \frac{2t}{1+t^2}P = \frac{1}{1+t^2}M$$

$$e^{\int \frac{2t}{1+t^2}dt} = e^{\ln(1+t^2)} = 1+t^2$$

$$\therefore P = \frac{1}{1+t^2}\int (1+t^2)\frac{1}{1+t^2}M\,dt$$

$$= \frac{1}{1+t^2}(Mt + C)$$

$$P(0)=0 \Rightarrow C = 0$$

$$\therefore P = \frac{t}{1+t^2}M.$$

$$P = 40\% M \Rightarrow \frac{2}{5} = \frac{t}{1+t^2}$$

$$\Rightarrow 2t^2 - 5t + 2 = 0$$

$$\Rightarrow (2t-1)(t-2) = 0$$

$$\Rightarrow \underline{\underline{t = \tfrac{1}{2}}} \text{ or } t = 2$$

Question 3 (b) [5 marks]

A certain bird population has a birth rate of 10% per year. They had been protected by law for many years and attained a logistic equilibrium of 100000 birds. The government then allowed people to shoot E birds per year and after a long time, the population settled down to a new equilibrium of 68000 birds. Find the value of E.

Answer 3(b)	2176

(Show your working below and on the next page.)

$$B_\infty = \frac{B}{S} \Rightarrow S = \frac{0.1}{100000} = \frac{1}{1000000}$$

$$\frac{dN}{dt} = BN - SN^2 - E = 0.1N - \frac{1}{1000000}N^2 - E$$

$$= -\frac{1}{1000000}\{N^2 - 100000N + 1000000E\}$$

Let $\beta_1 < \beta_2$ be the two roots of $N^2 - 100000N + 1000000E$.

$$\therefore \beta_2 = 68000$$

$$\because \beta_1 + \beta_2 = 100000 \Rightarrow \beta_1 = 100000 - 68000 = 32000$$

$$\therefore 1000000E = \beta_1\beta_2 = 32000 \times 68000$$

$$\therefore E = 32 \times 68 = \underline{\underline{2176}}$$

Question 4 (a) [5 marks]

A cantilevered beam of length L, made up of an extremely strong and light material, is horizontal at the end where it is attached to a wall, and carries a load of P Newtons at its end (that is, at $x = L$.) Assuming the weight of the beam is negligible compared to P, the beam has a moment function given by

$$M(x) = -(L - x)P.$$

Find the maximum deflection at $x = L$. (You may wish to recall the formula $EI = M/\frac{d^2y}{dx^2}$.)

Answer 4(a)	$-\dfrac{PL^3}{3EI}$

(Show your working below and on the next page.)

$$M(x) = EI \cdot \frac{d^2y}{dx^2} \Rightarrow \frac{d^2y}{dx^2} = -\frac{P}{EI}(L-x)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{P}{EI}\left(Lx - \frac{x^2}{2}\right) + C$$

Since $\frac{dy}{dx}(0) = 0 \Rightarrow C = 0$

$$\therefore \quad y = -\frac{P}{EI}\left(L\frac{x^2}{2} - \frac{x^3}{6}\right) + D$$

Since $y(0) = 0 \Rightarrow D = 0$

$$\therefore \quad y(x) = -\frac{P}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right)$$

$$y(L) = -\frac{P}{6EI}\left(3L^3 - L^3\right)$$

$$= -\frac{PL^3}{3EI}$$

Question 4 (b) [5 marks]

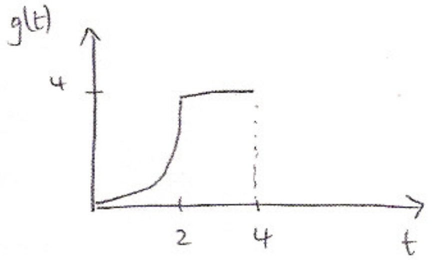
Let $g(t)$ be defined as

$$g(t) = \begin{cases} t^2, & 0 \leq t < 2 \\ 4, & 2 \leq t < 4 \\ 0, & t \geq 4. \end{cases}$$

Sketch $g(t)$ and compute its Laplace transform, $G(s)$, at $s = 4$.

Answer 4(b)	≈ 0.031156

(Show your working below and on the next page.)



$$g(t) = t^2\left(u(t) - u(t-2)\right) + 4\left(u(t-2) - u(t-4)\right)$$

$$= t^2 u(t) - (t^2-4)u(t-2) - 4u(t-4)$$

$$= t^2 u(t) - \left[(t-2)^2 + 4(t-2)\right]u(t-2) - 4u(t-4)$$

$$L(g(t)) = G(s) = \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2}\right) - 4e^{-4s}\cdot\left(\frac{1}{s}\right)$$

$$G(4) = \frac{2}{64} - e^{-8}\left(\frac{1}{32} + \frac{1}{4}\right) - e^{-16}$$

$$= \frac{2}{64} - \frac{9}{32}e^{-8} - e^{-16} \approx 0.031156$$

Question 5 (a) [5 marks]

Find the inverse Laplace transform of $\dfrac{1}{(s-1)(s^2+9)}$.

Answer 5(a)	$\frac{1}{10}e^{t} - \frac{1}{10}\cos 3t - \frac{1}{30}\sin 3t$

(Show your working below and on the next page.)

$$\frac{A}{s-1} + \frac{Bs+C}{s^2+9} = \frac{1}{(s-1)(s^2+9)}$$

$$\Rightarrow \quad As^2 + Bs^2 - Bs + Cs + 9A - C = 1$$

$$\Rightarrow \quad \left.\begin{array}{r} A+B=0 \\ -B+C=0 \\ 9A-C=1 \end{array}\right\} \Rightarrow \begin{array}{r} A+B=0 \\ 9A-B=1 \end{array}$$

$$\Rightarrow A = \frac{1}{10}, \quad B = -\frac{1}{10} = C.$$

$$\therefore L^{-1}\left(\frac{1}{(s-1)(s^2+9)}\right) = L^{-1}\left(\frac{1}{10(s-1)} - \frac{s+1}{10(s^2+9)}\right)$$

$$= \frac{1}{10}e^{t} - \frac{1}{10}\cos 3t - \frac{1}{30}\sin 3t \quad \#$$

Question 5 (b) [5 marks]

Solve the following initial value problem

$$y'' + 2\pi y' + 4\pi^2 y = \delta(t-1)$$

with initial conditions $y(0) = 1$ and $y'(0) = -\pi$. Evaluate $y(2)$.

Answer 5(b)	-0.0061333

(Show your working below and on the next page.)

$L(\delta(t-1)) = e^{-s}$

$$\therefore \quad Y(s) = L(y) = \frac{(s+2\pi)\cdot 1 - \pi + e^{-s}}{s^2 + 2\pi s + 4\pi^2}$$

$$= \frac{s+\pi + e^{-s}}{(s+\pi)^2 + 3\pi^2}$$

$$L^{-1}\left(\frac{s+\pi}{(s+\pi)^2+3\pi^2}\right) = e^{-\pi t}\cos(\sqrt{3}\pi t)$$

$$L^{-1}\left(\frac{e^{-s}}{(s+\pi)^2+3\pi^2}\right) = \frac{1}{\sqrt{3}\pi} e^{-\pi(t-1)} \sin(\sqrt{3}\pi(t-1))\, u(t-1)$$

$$\therefore y(t) = e^{-\pi t}\cos(\sqrt{3}\pi t) + \frac{1}{\sqrt{3}\pi} e^{-\pi(t-1)}\sin(\sqrt{3}\pi(t-1))\, u(t-1)$$

$$y(2) = e^{-2\pi}\cos(2\sqrt{3}\pi) + \frac{1}{\sqrt{3}\pi} e^{-\pi}\sin(\sqrt{3}\pi)$$

$$\approx -0.0061333$$

Question 6 (a) [5 marks]

Consider the following system of differential equations.

$$\frac{dx}{dt} = 2x + y, \qquad x(0) = 0,$$
$$\frac{dy}{dt} = x - 2y, \qquad y(0) = 1.$$

Find the solution for $x(t)$ using the Laplace transform.

Answer 6(a)	$\frac{\sqrt{5}}{5} \sinh(\sqrt{5}\, t)$
OR	$\frac{\sqrt{5}}{10} (e^{\sqrt{5} t} - e^{-\sqrt{5} t})$

(Show your working below and on the next page.)

$$L\left(\frac{dx}{dt}\right) = sX - 0 = 2X + Y \Rightarrow (s-2)X = Y$$

$$L\left(\frac{dy}{dt}\right) = sY - 1 = X - 2Y \Rightarrow (s+2)Y = X+1$$

Hence $(s+2)\big((s-2)X\big) = X+1$

$$\Rightarrow (s^2-4).X - X = 1$$

$$\Rightarrow X = \frac{1}{s^2-5} = \frac{1}{2\sqrt{5}}\left(\frac{1}{s-\sqrt{5}} - \frac{1}{s+\sqrt{5}}\right)$$

$$L^{-1}(X) = x(t) = \frac{\sqrt{5}}{10}\left(e^{\sqrt{5}t} - e^{-\sqrt{5}t}\right)$$

$$= \frac{\sqrt{5}}{5} \sinh(\sqrt{5}\, t)$$

Question 6 (b) [5 marks]

The weather of a typical day in Antartica is classified into a normal day, a cold day and an extremely cold day. There is a 10% chance that a normal day turns into a cold day and a 10% chance that a normal day turns into an extremely cold day. The probability that a cold day remains a cold day is 70% and the probability that a cold day turns into a normal day is $x\%$. An extremely cold day has a 90% chance of staying extremely cold and it is impossible for an extremely cold day to turn normal.

If today is extremely cold and the probability that three days from now is still extremely cold is 75.6%, find x.

Answer 6(b)	20%

(Show your working below and on the next page.)

$$M = \begin{array}{c} \begin{array}{ccc} N. & C & E \end{array} \\ \begin{bmatrix} 0.8 & x & 0 \\ 0.1 & 0.7 & 0.1 \\ 0.1 & 0.3-x & 0.9 \end{bmatrix} \end{array} \quad \text{and} \quad M^3 = \begin{bmatrix} & & \\ & & \\ & & 0.756 \end{bmatrix}$$

$$M^3 = \begin{bmatrix} 0.64+0.1x & 1.5x & 0.1x \\ 0.16 & 0.52 & 0.16 \\ 0.2-0.1x & 0.48-1.5x & 0.84-0.1x \end{bmatrix} \begin{bmatrix} 0.8 & x & \boxed{0} \\ 0.1 & 0.7 & \boxed{0.1} \\ 0.1 & 0.3-x & \boxed{0.9} \end{bmatrix}$$

$$\therefore \ (0.48-1.5x)0.1 + (0.84-0.1x)(0.9) = 0.756$$

$$0.048 - 0.24x = 0$$

$$x = \frac{0.048}{0.24} = 0.2$$

$$= 20\%$$

Question 7(a) [5 marks]

Given that the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

has three distinct eigenvectors which form a basis in three dimensions, and given that two of its eigenvalues are 0 and $\frac{1}{2}[3\sqrt{33}+15]$, find the third eigenvalue. Find also the eigenvector corresponding to the eigenvalue 0.

Answer 7(a)	$-\frac{1}{2}\left[3\sqrt{33}-15\right]$	$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

any non-zero multiple of this is acceptable.

(Show your working below and on the next page.)

Trace $= 1+5+9 = 15$ so if the unknown eigenv~~ector~~alue is λ,

$$\lambda + 0 + \frac{1}{2}\left[3\sqrt{33}+15\right] = 15 \Rightarrow$$

$$\lambda = -\frac{1}{2}\left[3\sqrt{33}-15\right]$$

Let $\begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}$ be the eigenvector corresponding to the zero eigenvalue; then

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} = 0 \Rightarrow \begin{cases} 1 + 2\alpha + 3\beta = 0 \\ 4 + 5\alpha + 6\beta = 0 \\ 2 + 4\alpha + 6\beta = 0 \end{cases}$$

$$\rightarrow 2 + \alpha = 0$$

$$\rightarrow \alpha = -2$$

$$\rightarrow 1 - 4 + 3\beta = 0$$

$$\rightarrow \beta = 1$$

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Question 7 (b) [5 marks]

A bioengineer studies the interactions of two kinds of bacteria in a particular culture. Bacterium A feeds on bacterium B and depends on it for its food, while bacterium B depends only on sunlight. The bioengineer checks the numbers of A and B every hour; let A_k and B_k be these numbers, measured in millions, in the k-th hour. His model of the situation is given by the following equations:

$$A_{k+1} = \frac{A_k}{2} + \frac{B_k}{100}, \quad B_{k+1} = -\frac{50A_k}{4} + \frac{5B_k}{4}.$$

Initially there are 50 million of type A and 5000 million of type B. By diagonalizing a matrix, compute how many bacteria of type A the model predicts there will be in four hours. [NOTE: zero marks if you don't diagonalize the matrix.]

Answer 7(b)	118·36 million

(Show your working below and on the next page.)

$$\begin{pmatrix} A_{k+1} \\ B_{k+1} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/100 \\ -\frac{50}{4} & 5/4 \end{pmatrix} \begin{pmatrix} A_k \\ B_k \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} A_4 \\ B_4 \end{pmatrix} = \begin{pmatrix} & \end{pmatrix}^4 \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} & \end{pmatrix}^4 \begin{pmatrix} 50 \\ 5000 \end{pmatrix}$$

Eigenvalues $\det \begin{pmatrix} \frac{1}{2} - \lambda & \frac{1}{100} \\ -\frac{50}{4} & \frac{5}{4} - \lambda \end{pmatrix} = 0$

$$\rightarrow \lambda^2 - \frac{7\lambda}{4} + \frac{3}{4} = 0 \rightarrow$$

$$\lambda = \frac{1}{2}\left[\frac{7}{4} \pm \sqrt{\frac{49}{16} - 3} \right] = 1, \ 3/4$$

(More working space for Question 7(b))

Eigenvector for 1 is

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{100} \\ -\frac{50}{4} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 1 \\ \alpha \end{pmatrix} = 0 \rightarrow -\frac{1}{2} + \frac{\alpha}{100} = 0$$

$$\rightarrow \alpha = 50$$

Eigenvector for $3/4$ is

$$\begin{pmatrix} -1/4 & \frac{1}{100} \\ -\frac{50}{4} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ \alpha \end{pmatrix} = 0 \rightarrow \alpha = 25$$

$$P = \begin{pmatrix} 1 & 1 \\ 50 & 25 \end{pmatrix} \rightarrow P^{-1} = \begin{pmatrix} -1 & +1/25 \\ 2 & -1/25 \end{pmatrix}$$

$$(\quad)^4 = \begin{pmatrix} 1 & 1 \\ 50 & 25 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (3/4)^4 \end{pmatrix}\begin{pmatrix} -1 & 1/25 \\ 2 & -1/25 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 50 & 25 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{81}{256} \end{pmatrix}\begin{pmatrix} -1 & 1/25 \\ 2 & -1/25 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 50 & 25 \end{pmatrix}\begin{pmatrix} -1 & 1/25 \\ \frac{81}{128} & -\frac{81}{6400} \end{pmatrix} = \begin{pmatrix} \frac{81}{128} - 1 & \frac{1}{25} - \frac{81}{6400} \\ * & * \end{pmatrix}\begin{pmatrix} 50 \\ 5000 \end{pmatrix}$$

$$A_4 = 50\left(\frac{81}{128} - 1 + \frac{100}{25} - \frac{8100}{6400}\right) = 118.36 \text{ million}$$

Question 8 (a) [5 marks]

A chemical engineer has two tanks containing 100 litres of water. Tank A initially contains water in which 25 kg of a dangerous chemical are dissolved, and tank B contains x kilograms of this chemical. Pure water is poured into tank A at a constant rate of 4 litres per minute. The thoroughly mixed solution from tank A is constantly pumped into tank B at a rate of 6 litres per minute, while the solution from tank B is pumped back to tank A at a rate of 2 litres per minute. The solution in tank B is also pumped out and discarded at a rate of 4 litres per minute. The engineer wants to choose x in such a way that the ratio of the amount of the chemical in tank A to the amount in tank B is constant. Find x.

Answer 8(a)	$25\sqrt{3}$ Kg ≈ 43.3 Kg

(Show your working below and on the next page.)

Let x_A, x_B be the number of kgs of the chemical in tanks A, B. Then

$$\dot{x}_A = \frac{1}{100}(-6x_A + 2x_B)$$

$$\dot{x}_B = \frac{1}{100}(6x_A - 6x_B)$$

so

$$\begin{pmatrix} \dot{x}_A \\ \dot{x}_B \end{pmatrix} = \frac{1}{100}\begin{pmatrix} -6 & 2 \\ 6 & -6 \end{pmatrix}\begin{pmatrix} x_A \\ x_B \end{pmatrix}.$$

Since the trace is $-12/100$, det $= \frac{24}{(100)^2}$,

$(\text{Trace})^2 - 4\det = \frac{144}{100^2} - \frac{4 \times 24}{100^2} > 0$ so we have a nodal sink.

The only way x_A/x_B can be constant is if the trajectory in the phase plane is a straight line, which is only possible if that line is

(More working space for Question 8(a))

parallel to an eigenvector in the first quadrant.

$$\det \begin{bmatrix} -\frac{6}{100} - \lambda & \frac{2}{100} \\ \frac{6}{100} & -\frac{6}{100} - \lambda \end{bmatrix} = 0 \rightarrow \lambda^2 + \frac{12}{100}\lambda + \frac{24}{10000} = 0$$

$$\lambda = \frac{1}{2}\left[-\frac{12}{100} \pm \sqrt{\frac{144}{10000} - \frac{96}{10000}} \right] = \frac{-6 \pm 2\sqrt{3}}{100}$$

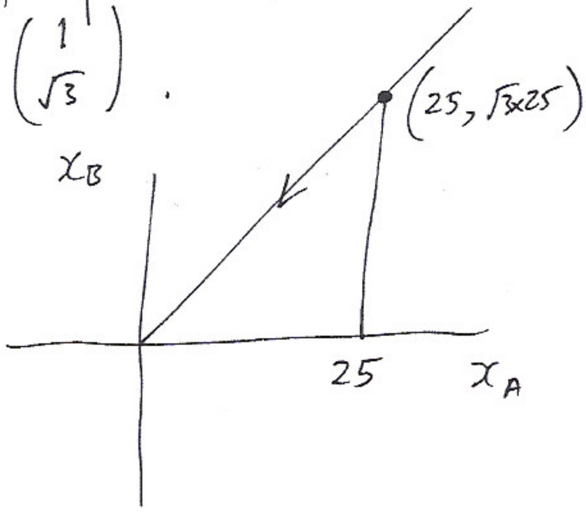
Eigenvectors:

$$\begin{bmatrix} \pm \frac{2\sqrt{3}}{100} & \frac{2}{100} \\ * & * \end{bmatrix} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = 0 \rightarrow \alpha = \pm\sqrt{3}$$

Since we want the first quadrant, we are interested in the vector $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$.

The line has the equation

$$x_B = \sqrt{3}\, x_A$$



so the amount of chemical initially in tank B should be $\sqrt{3} \times 25$

$$\approx 43.30 \text{ kg}$$

Question 8(b) [5 marks]

Classify the systems of linear ordinary differential equations with the following coefficient matrices:

(a) $\begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$; (b) $\begin{pmatrix} 2 & -2 \\ 7 & 0 \end{pmatrix}$; (c) $\begin{pmatrix} -2 & -8 \\ 5 & 0 \end{pmatrix}$, (d) $\begin{pmatrix} -6 & 4 \\ -2 & 1 \end{pmatrix}$, (e) $\begin{pmatrix} 7 & -4 \\ 2 & -1 \end{pmatrix}$.

Answer 8(b)	

(Show your working below and on the next page.)

(a) $\begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$ Trace $= -1$ det $= -4$ $\rightarrow$ SADDLE

(b) $\begin{pmatrix} 2 & -2 \\ 7 & 0 \end{pmatrix}$ Trace $= 2$ det $= 14$, $Tr^2 - 4det = 4 - 56 < 0$

$\rightarrow$ SPIRAL SOURCE

(c) $\begin{pmatrix} -2 & -8 \\ 5 & 0 \end{pmatrix}$ Trace $= -2$ det $= 40$ $Tr^2 - 4det < 0$

$\rightarrow$ SPIRAL SINK

(d) $\begin{pmatrix} -6 & 4 \\ -2 & 1 \end{pmatrix}$ Trace $= -5$ det $= 2$ $Tr^2 - 4\,det > 0$

$\rightarrow$ NODAL SINK

(e) $\begin{pmatrix} 7 & -4 \\ 2 & -1 \end{pmatrix}$ Trace $= +6$ det $= 1$ $Tr^2 - 4det > 0$

$\rightarrow$ NODAL SOURCE