1. (a) A subset $K$ of a metric space $X$ is said to be compact if every open cover of $K$ contains a finite subcover.

Prove that compact subsets of metric spaces are closed.

(b) A subset $E$ of a metric space $X$ is said to be perfect if $E$ is closed and if every point of $E$ is a limit point of $E$.

Prove that if $E$ is a non-empty perfect set of $\mathbb{R}$. Then $E$ is uncountable.

(c) Prove that the open interval $(a, b)$ is uncountable.

2. (a) Let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous, $n = 1, 2, \ldots$. Suppose that

$$f(x) = \sum_{u=0}^{\infty} f_n(x)$$

exists for every $x \in \mathbb{R}$.

Is $f : \mathbb{R} \to \mathbb{R}$ continuous? Justify your answer.

(b) Suppose $f_n \to f$ uniformly on a set $E$ in a metric space. Let $x$ be a limit point of $E$ and suppose that

$$\lim_{t \to x} f_n(t) = A_n, \ n = 1, 2, \ldots .$$

Prove that $\lim_{n \to \infty} A_n$ exists and

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t).$$

(c) Let $\{f_n\}$ be a sequence of continuous functions on $(0, 1)$ such that $\{f_n\}$ converges pointwise to a continuous function on $(0, 1)$ and $f_n(x) \geq f_{n+1}(x)$ for all $x \in (0, 1)$, $n = 1, 2, \ldots$.

Does $\{f_n\}$ converge uniformly to $f$ on $(0, 1)$? Justify your answer.
3. Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$, let $\alpha > 0$. If there exists a constant $k > 0$ such that

$$|f(x) - f(y)| \leq k|x - y|^\alpha$$

for all $x, y \in A$, then $f$ is said to be a Lipschitz function of order $\alpha$ on $A$.

(a) Suppose $f$ is a Lipschitz function of order $\alpha$ on $(0, 1)$ where $\alpha > 1$. Prove that $f$ is differentiable on $(0, 1)$ and find its derivative $f'$.

(b) Give an example of a Lipschitz function of order $\frac{1}{2}$ but not of order 1 on $[0, 1]$.

(c) Is every uniformly continuous function on $[0, 1]$ is a Lipschitz function of order 1? Justify your answer.

4. (a) Let $f_n : [0, 1] \to \mathbb{R}$ be continuous, $n = 1, 2, \ldots$. Suppose $\{f_n\}$ converges uniformly on $[0, 1]$. Prove that $\{f_n\}$ is equicontinuous on $[0, 1]$.

(b) Let $f_n : [0, 1] \to \mathbb{R}$ be continuous, $n = 1, 2, \ldots$. Suppose $\{f_n\}$ is pointwise bounded and equicontinuous on $[0, 1]$. Prove that (i) $\{f_n\}$ is uniformly bounded on $[0, 1]$; (ii) $\{f_n\}$ contains a uniformly convergent subsequence.

END OF PAPER
INSTRUCTIONS TO CANDIDATES

Answer ALL questions.