NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2005-2006

MA5205 Graduate Analysis I

November/December 2005 — Time allowed : $2\frac{1}{2}$ hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **FIVE (5)** questions and comprises **FOUR (4)** printed pages.
2. Answer as many questions as you can.
3. Each question carries the same weight.

Question 1

(a) If E is a subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, define

$$x + E = \{x + y : y \in E\}.$$

Show that if E is a Lebesgue measurable set, then so is $x + E$.

(b) Let $(\mathbb{R}, \Sigma, \lambda)$ and $(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu)$ be the Lebesgue measure space and the counting measure space on $\mathbb{R}$ respectively. Show that for any subset E of $\mathbb{R}$ and any $y \in \mathbb{R}$

$$(\lambda \times \mu)^*(E \times \{y\}) = \lambda^*(E).$$

Question 2

Let (Ω, Σ, μ) be a σ-finite, complete measure space and let λ and Σ_λ be the Lebesgue measure and the σ-algebra of Lebesgue measurable sets respectively. If f is a nonnegative extended real-valued μ-measurable function on Ω and $1 < p < \infty$, define

$$g(x) = \begin{cases} x^{p-2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$h(x) = \begin{cases} \int f \chi_{\{f \geq x\}} \, d\mu & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\int f^p \, d\mu = (p-1) \int g(x) h(x) \, d\lambda(x).$$

[Use (...) Theorem (fill in the blank). Be sure to check the hypotheses of the theorem used.]

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Question 3

Let (Ω, Σ, μ) be a measure space such that $\mu(\Omega) < \infty$. Suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of extended real-valued integrable functions with the following properties.

(i) $\sup_n \int |f_n|\, d\mu < \infty$,

(ii) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\sup_n \int |f_n| \chi_E \, d\mu < \varepsilon$ whenever $E \in \Sigma$, $\mu(E) < \delta$.

(a) Show that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup_n \int |f_n| \chi_{\{|f_n| \geq N\}} < \varepsilon.$$

For any extended real-valued function g defined on Ω and any $N \in \mathbb{N}$, define g^N on Ω by

$$g^N(\omega) = \begin{cases} g(\omega) & \text{if } |g(\omega)| < N \\ N & \text{if } g(\omega) \geq N \\ -N & \text{if } g(\omega) \leq -N. \end{cases}$$

From here on, assume in addition that $(f_n)_{n=1}^{\infty}$ converges to a function f pointwise.

(b) Show that for all $N \in \mathbb{N}$, $(f_n^N)_{n=1}^{\infty}$ converges pointwise to f^N.

(c) Show that $\lim_n \int f_n \, d\mu = \int f \, d\mu$.

Question 4

Show that a bounded real-valued function that is Riemann integrable on a bounded real interval $[a,b]$ is Lebesgue integrable on $[a,b]$ and that the values of the integrals are equal.

To be definite, I include the definition of the Riemann integral. Given a bounded interval $[a,b]$ in $\mathbb{R}$, a *partition* of $[a,b]$ is a finite sequence of points $P = (p_i)_{i=0}^{n}$ satisfying $a = p_0 < \cdots < p_n = b$. If f is a bounded real-valued function on $[a,b]$ and P is a partition of $[a,b]$, the *upper* and *lower sums* of f with respect to P are

$$U(f,P) = \sum_{i=1}^{n} \sup_{x \in [p_{i-1},p_i]} f(x)(p_i - p_{i-1}) \quad \text{and}$$
$$L(f,P) = \sum_{i=1}^{n} \inf_{x \in [p_{i-1},p_i]} f(x)(p_i - p_{i-1}).$$

f is *Riemann integrable* if there exists $R \in \mathbb{R}$ such that

$$\begin{aligned}\inf\{U(f,P) : P \text{ is a partition of } [a,b]\} &= R \\ &= \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}.\end{aligned}$$

In this case, we call R the *Riemann integral* of f on $[a,b]$ and denote it by $(R) - \int_a^b f$.

[The upper and lower sums correspond naturally to simple functions.]

Question 5

Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. Define $V : [a,b] \to \mathbb{R}$ by $V(a) = 0$ and $V(x) = \mathrm{Var}_{[a,x]} f$ for $a < x \leq b$. If $\int_a^b |f'| \, d\lambda = \mathrm{Var}_{[a,b]} f$, show that V is absolutely continuous and hence so is f.

END OF PAPER