NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2003-2004

MA5213 Advanced Partial Differential Equations

November 2003 — Time allowed : 5 days

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **ONE (1)** section. It contains **FIVE (5)** questions and comprises **Two (2)** printed pages.

2. Answer **ALL** questions. Each question carries 20 marks.

3. Candidates may use calculators and any references they can find. However, they should lay out systematically the various steps in the calculations, as well as the reasoning just based on the course.

Question 1

(a) Find all the solutions of the equation $\Delta v + k|\nabla v|^2 = e^{-kv}$ in the unit ball of R^n with the zero Dirichlet boundary condition and $n \geq 3$ and $k \neq 0$.

(b) Show that if u and v are real-valued harmonic functions in $\Omega \subset R^n$, then uv is harmonic in Ω if and only if $\nabla u \cdot \nabla v \equiv 0$.

Question 2

(a) Suppose $\Omega \subset R^n$ is a bounded set and $\partial\Omega$ is connected set. Show that if u is a real-valued continuous function on $\bar{\Omega}$ that is harmonic, then $u(\Omega) \subset u(\partial\Omega)$.

(b) Show that for all $y \in R^n$ with $|y| = 1$, $|x||x^* + y| = |x + y|$ for all $x \in R^n \backslash \{0\}$ where $x^* = x/|x|^2$.

Question 3

Suppose Ω is bounded and connected, u is a positive harmonic function on Ω, and α is a multi-index. Show that there is a constant $C > 0$ such that

$$|\frac{\partial^\alpha u}{\partial x^\alpha}(a)| \leq \frac{C}{d(a, \partial\Omega)^{|\alpha|+n-1}}$$

for all $a \in \Omega$.

Question 4

Construct suitable Banach spaces B_1, B_2 as function spaces over a domain $\Omega \subset R^n$ such that $T = \Delta : B_1 \longrightarrow B_2$ is a compact linear mapping.

Question 5

Use the Lax-Milgram theorem to show that for any functions $f, g \in H^1(B_R(0)) := \{v | v \in L^2(B_R(0)), |\nabla v| \in L^2(B_R(0))\}$, there exists a constant $\bar{\mu} > 0$ such that if $\mu > \bar{\mu}$, the functional

$$J(u) = \int_{B_R(0)} \left[|\nabla u|^2 + \mu u^2\right] dx - \int_{B_R(0)} f u dx$$

has a root on the set $M = \{u \in H^1(B_R(0)) | u = g \;\; on \;\; \partial B_R(0)\}$ where ∇ is the gradient operator.

END OF PAPER