PH.D. QUALIFYING EXAMINATION 2003/2004 (Sem 2)
ALGEBRA

1. Show that if a and b are elements in a group G, then ab and ba have the same order. [10 marks]

2. (a) Let H and K be subgroups of a group G with H normal in G. Show that
$$HK := \{hk : h \in H, k \in K\}$$
is a subgroup of G and show that H is normal in HK. [10 marks]

   (b) Show that $(H \cap K)$ is normal in K and that
$$K/(H \cap K) \simeq HK/H.$$
[10 marks]

   (c) Show that if H is a normal subgroup of G such that
$$\gcd(|H|, [G : H]) = 1,$$
then H is the unique subgroup of G of order $|H|$.
[15 marks]

3. (a) Show that if R is a finite integral domain with a unit element, then R is a field. [10 marks]

   (b) Show that if R is a finite commutative ring with a unit element, then every prime ideal of R is a maximal ideal. [10 marks]

4. Let R is a ring with a unit element, 1_R, in which
$$(ab)^2 = a^2b^2$$
for all $a, b \in R$. Prove that R must be commutative. [15 marks]

5. (a) Let K be a finite field of p elements, where p is a prime. Let $\gcd(n, p) = 1$ and F be the splitting field of $x^n - 1_K$ over K. Show that if $(F : K) = f$ then n divides $q^f - 1$.
[10 marks]

   (b) Show that f is the smallest integer m for which $q^m - 1$ is divisible by n.
[10 marks]

END OF PAPER