Ph.D Qualifying Examination
Linear Algebra

(1) State and prove the Cayley-Hamilton theorem for a 5 by 5 real matrix A.

(2) Let X be a 5 by 5 complex matrix. Let $\mathbf{v}_1, \ldots, \mathbf{v}_5$ be nonzero eigenvectors of X with distinct eigenvalues. Prove that $\mathbf{v}_1, \ldots, \mathbf{v}_5$ are linearly independent over $\mathbb{C}$. Show that X is diagonalizable.

(3) Suppose A and B are invertible real matrices. Show that

$$\mathrm{adj}(AB) = \mathrm{adj}(B)\mathrm{adj}(A).$$

Is the above true if both A and B are not invertible matrices? If it is true give a proof. If it is false, give a counter example.

(4) Prove that the row rank and the column rank of an 8 by 10 matrix X are the same.

(5) Let X be a 5 by 5 symmetric real positive definite matrix (that is $\mathbf{v}^T X \mathbf{v} > 0$ for all nonzero column vectors $\mathbf{v} \in \mathbb{R}^5$). Prove that there exists an invertible upper triangular 5 by 5 matrix P such that

$$X = P^T P.$$

(6) Let

$$X = \frac{1}{11}\begin{pmatrix} -6 & 28 & 14 & -3 \\ -10 & 32 & 5 & 6 \\ 5 & -5 & 25 & -3 \\ -25 & 25 & 18 & 4 \end{pmatrix}.$$

It is known that it has two only eigenvalues, namely 2 and -1. Find the characteristic polynomial $p(x)$, minimal polynomial $m(x)$ and Jordan canonical form D of X. Find a matrix P such that

$$X = PDP^{-1}$$

(You do not have to find P^{-1}.)

(7) Let $\langle\, , \rangle$ be an inner product on $\mathbb{R}^5$. Let X be a 5 by 5 matrix such that the columns are orthogonal to each other and have length 1 with respect to $\langle\, , \rangle$.

(a) Suppose $\langle\, , \rangle$ is the usual Euclidean dot product. Show that the rows are are orthogonal to each other and have length 1 with respect to $\langle\, , \rangle$.

(b) Suppose $\langle\, , \rangle$ is the NOT the usual Euclidean dot product. Define an inner product $\langle\, , \rangle'$ on $\mathbb{R}^5$ using $\langle\, , \rangle$ such that the rows are orthogonal to each other and have length 1 with respect to $\langle\, , \rangle'$.

(8) Let X_1, X_2 be two diagonalizable 5 by 5 matrices. Suppose the matrices commute with each other, that is, $X_1X_2 = X_2X_1$. Show that the two matrices are simultaneously diagonalizable, that is, there exists a 5 by 5 matrix invertible P such that $P^{-1}X_1P$ and $P^{-1}X_2P$ are diagonal matrices.

END OF PAPER