Ph.D. Qualifying Examination

Algebra

1. (a) Let G be the additive group $\mathbb{Q}/\mathbb{Z}$. Show that any finite subgroup of G is cyclic.

   (b) For the ring $R = \mathbb{Z} \times \mathbb{Z}$, give an example for each of the following:

   (i) a maximal ideal of R;

   (ii) a prime ideal of R that is not maximal.

2. (a) Let G be a finite group, and H be a subgroup of index 2. Show that $x^2 \in H$ for any $x \in G$ and hence deduce that H contains all elements of G of odd order.

   (b) Let $n > 3$ be an integer, and let G be a subgroup of S_n. Assume that G has an odd permutation. Show that G has a normal subgroup of index 2.

   (c) Let A_4 be the subgroup of even permutations in S_4. Show that A_4 has no subgroup of index 2.

3. Recall that an element p of an integral domain D is called irreducible if p is a non-zero, non-unit and in any factorization $p = rs$ with $r, s \in D$, one of r, s is a unit. Now let

$$D = \mathbb{Z}[\sqrt{-7}] = \{a + b\sqrt{-7} | a, b \in \mathbb{Z}\}.$$

   (i) By using the norm function $N(a + b\sqrt{-7}) = a^2 + 7b^2$, show that $2, 1 \pm \sqrt{-7}$ are irreducible elements of D.

   (ii) Is $2D$ a prime ideal? Is D a unique factorization domain? Justify your answers.

4. (a) Let R be a finite commutative ring with 1, such that $1 \neq 0$. Let $R^* = R \backslash \{0\}$ and put

$$k = \prod_{r \in R^*} r,$$

   is a field.

   (b) Let p be a positive prime number such that $p = 4k + 1$ for some $k \in \mathbb{Z}$. Show that there exists $a \in \mathbb{Z}_p$ such that $a^2 = -1$ in $\mathbb{Z}_p$.

5. Show that each of the following polynomials is irreducible over $\mathbb{Q}$: you may want to consider reduction modulo a prime number.

(i) $3x^4 - 2x^2 + 72x - 10$;

(ii) $x^3 + 1003x + 1002$.

END OF PAPER