NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2002-2003

MA5213 Advanced Partial Differential Equations

October/November 2002 — Time allowed : 5 days

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **ONE (1)** section. It contains **FIVE (5)** questions and comprises **Two (2)** printed pages.

2. Answer **ALL** questions. Each question carries 20 marks.

3. Candidates may use calculators and any references they can find. However, they should lay out systematically the various steps in the calculations, as well as the reasoning just based on the course.

Question 1

(a) Find all the solutions of the equation $\Delta v + |\nabla v|^2 = \frac{n}{r^2}$ in the unit ball of R^n with the zero Dirichlet boundary condition and $n \geq 3$.

(b) Show that if $\lambda \leq 0$, then the equation

$$\Delta u + \lambda u + u^{\frac{n+2}{n-2}} = 0 \quad \text{in } B_1(0) \subset R^n, n \geq 3,$$

with the boundary condition, $u = 0$ on $\partial B_1(0)$, has no positive solution.

Question 2

Suppose u satisfies the equation $\Delta u = f$ in the ball $B_R(0) \subset R^n$ with zero Dirichlet boundary value. Show the following estimates:
(i) $\sup_{B_R(0)} |u| \leq \frac{R^2}{2n} \sup_{B_R(0)} |f|$;

(ii) $\sup_{B_R(0)} |\nabla u| \leq R \sup_{B_R(0)} |f|$.

Question 3

Find a continuous function f on R^n such that the Poisson equation $\Delta u = f$ has no C^2 solution in any neighbourhood of the origin.

Question 4

Show that if the function $u \in H^2(R^4)$, there exists a constant $\gamma = \gamma(u) > 0$ such that

$$\int_{R^4} (e^{\gamma |u|^2} - 1) dx < \infty.$$

Question 5

Use the Lax-Milgram theorem to show that for any functions $f, g \in H^1(B_R(0))$, there exists a constant $\bar{\mu} > 0$ such that if $\mu > \bar{\mu}$, then the non-homogenous Dirichlet problem $-\Delta u + \mu u = f$ in $B_R(0)$ with $u = g$ on the boundary of the ball has a unique weak solution.

END OF PAPER